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Logical stochastic resonance in bistable system under α-stable noise

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Abstract. In the presence of α-stable noise, the logical stochastic resonance (LSR) phenomenon in a class of double well nonlinear system is investigated in this paper. LSR effect is obtained under α-stable noise. The probability of getting correct logic outputs is used to evaluate LSR behavior. Four main results are presented. Firstly, in the optimal band of noise intensity, Gaussian white noise is considered a better choice than heavy tailed noise to obtain clean logic operation. But at weak noise background, the success probability of getting the right logic outputs is higher when the system is subjected to heavy tailed noise. Secondly, it is shown that over the entire range of noise variance, the asymmetric noise induced LSR performs better than that induced by the symmetric noise. Furthermore, we find which side the tail skews also affects the correct probability of LSR. At last, the fractional Fokker-Planck equation is presented to show when the characteristic exponent of α-stable noise is less than 1, LSR behavior will not be obtained irrespective of the setting for other parameters.

1 Introduction

Stochastic resonance (SR) is a noise-induced phenomenon that demonstrates the interplay of noise and nonlinear system will amplify and optimize the feeble input signal. Benzi et al. [1–3] pioneered the concept in one of their seminal papers which addressed the problem of the periodically recurrent ice ages. After that, SR has been found in a large number of areas, ranging from optical, electronic to magnetic systems. Hänggi extended SR to biology realm, and investigated how noise can enhance detection of weak signals and help improve biological information processing [4]. Gammaitoni et al. [5] have written an extensive review for the classical SR theory and its importance applications.

Recently, Murali et al. [6,7] introduced the concept of logical stochastic resonance (LSR) wherein the output of a bistable dynamical system can be a logical combination of the two input signals when the system is driven by an optimal band of Gaussian white noise. The study pointed out an intriguing possibility in the realization of new paradigm of integrated circuits to deal with the tighter noise margins. As an emerging concept, LSR has attracted a lot of attention from different research groups. Animesh et al. found that dynamical behavior equivalent to LSR can also be obtained without noise [8]. Remo et al. [9] extended the study of LSR from the bistable system to the multistable (tri-stable) system given by piecewise functions and obtained XOR logic. Kohar and Sinha [10] demonstrated that in the optimal range of noise, the asymmetric bistable fourth order system behaves like a memory device as well as a logic device. Dari et al. [11,12] and Hellen et al. [13] studied the LSR phenomenon in synthetic gene network.

According to the central limit theorem, the noise in physical systems is usually considered to be the standard Gaussian distribution. Nevertheless, the empirical evidence suggests that there is a need to consider various types of noise. Zhang et al. [14] exploited the LSR phenomenon in a class of triple-well systems induced by additive or multiplicative Gaussian colored noise. Zhang et al. [15,16] investigated the effects of Ornstein-Uhlenbeck noise and $1/f$ noise on LSR. The dichotomous noise induced LSR in energetic and entropic systems is studied by Das and Ray in reference [17].

As the generalization of Gaussian noise, α-stable noise (also known as Lévy stable noise) has often been used to describe the impulsive characteristic of noise in real-world applications. There has recent been more attention on nonlinear system driven by α-stable noise. The related studies include M-ary signal detection via a bistable system in the presence of Lévy noise [18], multiplicative Lévy noise in bistable systems [19], and evaluation of an asymmetric bistable system for signal detection under Lévy stable noise [20]. Conventional SR is proved can be obtained and detected by common quantifiers in presence of α-stable noise by Dybiec and Gudowska-Nowak [21,22]. Szczepaniec and Dybiec [23] demonstrated that non-equilibrium α-stable noises, depending on noise parameters, can either weaken or enhance the non-dynamical stochastic resonance. Tang et al. [24]

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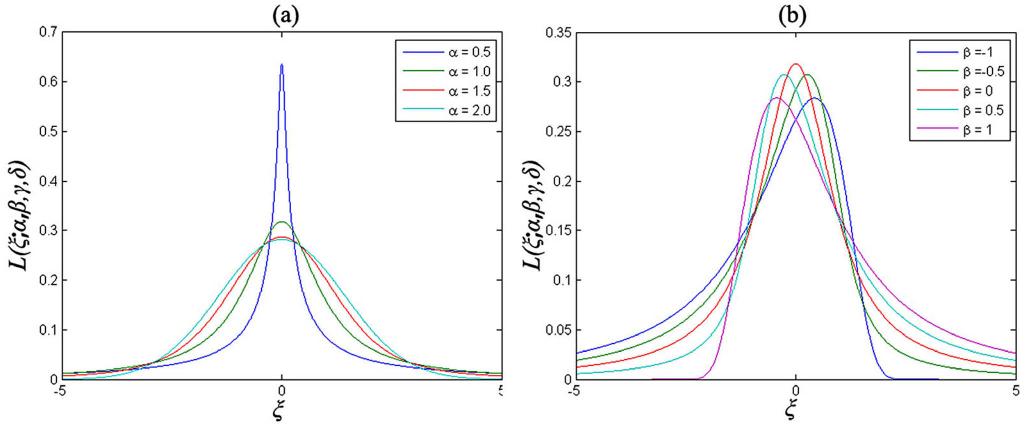

Fig. 1. Probability density function of α-stable distribution L (ξ; α, β, γ, δ). (a) Varying α when $\beta = 0$, $\gamma = 1$, $\delta = 0$; (b) varying β when $\alpha = 1$, $\gamma = 1$, $\delta = 0$.

investigated stochastic resonance in an ensemble of bistable systems under stable distribution noises and nonhomogeneous coupling. Xu et al. [25] characterized the influences of the intensity and stability index of Lévy noise, as well as the amplitude of external signal on the occurrence of stochastic resonance phenomenon. However, LSR behavior under α-stable noise has never been reported in the open literature.

In this paper, we study the LSR phenomenon in double-well system subjected to α-stable noise. How the characteristic of α-stable noise affects LSR is analyzed and numerical stimulated explicitly. The outline of the paper is organized as follows: a bistable system depicted by piecewise function and α-stable noise is described in Section 2. Section 3 presents the numerical stimulation and discussion of α-stable noise induced LSR. We close this work with some final remarks and conclusions in Section 4.

2 Models and measurements

Considering an overdamped Brownian particle in a very simple nonlinear system [7], the Langevin-type equation is given by

$$\dot{x} = -ax + bg(x) + \xi(t) + I(t), \qquad (1)$$

where a and b are the nonlinear parameters, $g(x)$ is a piecewise function with the form:

$$g(x) = \begin{cases} x_l & x < x_l \\ x & x_l \leqslant x \leqslant x_u \\ x_u & x > x_u, \end{cases} \qquad (2)$$

and x_l, x_u are the lower and upper thresholds, respectively.

$\xi(t)$ is α-stable noise. The α-stable distribution is a four-parameter family of distributions denoted by L (ξ; α, β, γ, δ). The first parameter $\alpha \in (0, 2]$ is called the characteristic exponent, which describes the tail of the distribution. The skewness parameter $\beta \in [-1, 1]$ specifies whether the distribution is right ($\beta > 0$) or left ($\beta < 0$) skewed. The last two parameters are the scale $\gamma > 0$, and the location $\delta \in R$. $D = \gamma^\alpha$ represents the noise intensity. One of the nice properties of the α-stable distribution is that it generalizes the classical central limit theorem. The family of α-stable distribution includes the following distributions as subclasses: Gaussian distribution with $\alpha = 2$, Cauchy distribution with $\alpha = 1$, $\beta = 0$ and Lévy distribution with $\alpha = 1/2$, $\beta = 1$. Except for these three cases, the density function of α-stable random variable cannot be given in closed form. However, the characteristic function can be given as follows:

$$\begin{aligned}\varphi(t) &= E\exp(it\xi) \\ &= \begin{cases} \exp\left(-\gamma^\alpha |t|^\alpha \left[1 - i\beta \mathrm{sign}(t)\tan\left(\frac{\pi\alpha}{2}\right)\right] + i\delta t\right) & \alpha \neq 1 \\ \exp\left(-\gamma |t| \left[1 + i\beta \mathrm{sign}(t)\frac{2}{\pi}\log|t|\right] + i\delta t\right) & \alpha = 1. \end{cases}\end{aligned} \qquad (3)$$

In this paper, the Janicki-Weron algorithm [26] is employed to generate the α-stable distribution random numbers. With no loss of generality, we assume the location $\delta = 0$. Figure 1 depicts the typical probability density function of the α-stable distribution L (ξ; α, β, γ, δ).

$I(t)$ is composed by two trains of aperiodic pulses: $I_1(t)$ and $I_2(t)$, with $I_1(t)$ and $I_2(t)$ encoding the two logic inputs. Since the logical value of $I_1(t)$ and $I_2(t)$ can be either 0 or 1, the combination of these two inputs can be (0,0), (0,1), (1,0) and (1,1). And the input sets (0,1) and (1,0) give rise to the same $I(t)$.

The logic output of the system is determined by its state, e.g., the output can be considered a logical 1 (or 0) when it is in one well x_+ (or x_-), and logical 0 (or 1) when it is in the other well x_- (or x_+). For a given input set (I_1, I_2), the logical operation of the system can be checked based on the true Table 1. In this paper, we value $a = 1.8$, $b = 3$, $x_u = 1.3$, $x_l = -0.5$ to get OR (NOR) logic, and $a = 1.8$, $b = 3$, $x_u = 0.5$, $x_l = -1.3$ to get AND (NAND) logic. For the convenience of numerical computation, we consider the two inputs $I_1(t)$ and $I_2(t)$ to take a value of −0.5 when it is considered to be logical 0, and a value of 0.5 when it is considered to be logical 1.

Table 1. Relationship between two logic inputs and the logic output of the four fundamental AND, NAND, OR, and NOR gates. The combination of the four fundamental logic gates can be used to construct any logic circuit.

Input set (I_1, I_2)	OR	NOR	AND	NAND
(0,0)	0	1	0	1
(0,1)/(1,0)	1	0	0	1
(1,1)	1	0	1	0

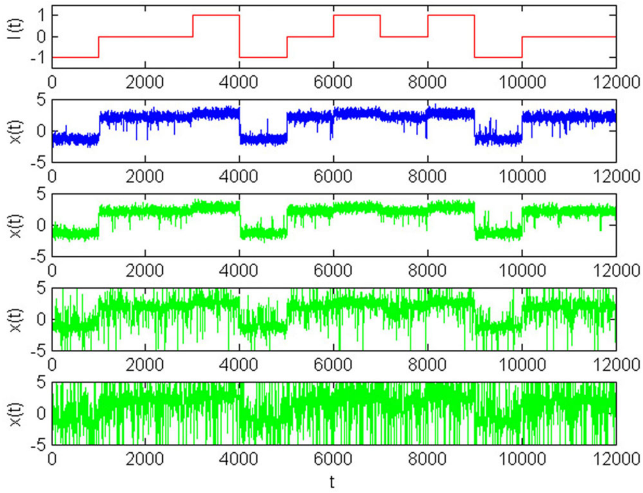

Fig. 2. Panels from top to bottom, the input signal $I(t)$, response of the system to Gaussian noise, responses to α-stable distribution noise with $\alpha = 2$, 1.5, 1, respectively. Here, the intensity of Gaussian noise $D = 0.8$, $\beta = 0$, $\gamma = D/\sqrt{2}$.

3 Simulations

3.1 Effect of characteristic exponent α on LSR

Firstly, we consider the effect of characteristic exponent α on LSR. Characteristic exponent α describes the tail of the distribution. When $\alpha = 2$, the distribution converges to Gaussian distribution with variance $\sigma^2 = 2\gamma^2$. When $\alpha < 2$, α specifies the asymptotic behavior of the distribution. With the decrease of α, three changes occur to the density: the peak turns higher, the region flanking the peak turns lower, and the tails turn heavier (Fig. 1a).

Figure 2 shows the responses of the system (1) subjected to Gaussian white noise and α-stable noise with different characteristic exponents, respectively. It can be seen that the α-stable noise can also induce LSR phenomenon in nonlinear system.

To show more directly, we calculate the probability of correct logic output, P, to evaluate the LSR effect. Modified Murali et al. [7] experiment setups, P is obtained as follows: use the random permutated combination of 4 logical inputs sets (0,0), (0,1), (1,0) and (1,1) to drive the system over some reasonably time τ for N runs, with each input set driving for $\tau/4$. In each run, sample $x(t)$ for the four input sets. Calculate the correct probability P_i, the ratio of correct logical output number of sampled $x(t)$ to the total number of sampled $x(t)$, in each run. Specially, in each single input set, the calculation error of $1 \sim 3\%$, which is caused by the relaxation time, is acceptable. In order to reduce the effect of relaxation time, we sample $x(t)$ in different input set with a reasonable time delay. In this strategy, the probability P is given by

$$P = \frac{\sum_{i=1}^{N} p_i}{N}. \tag{4}$$

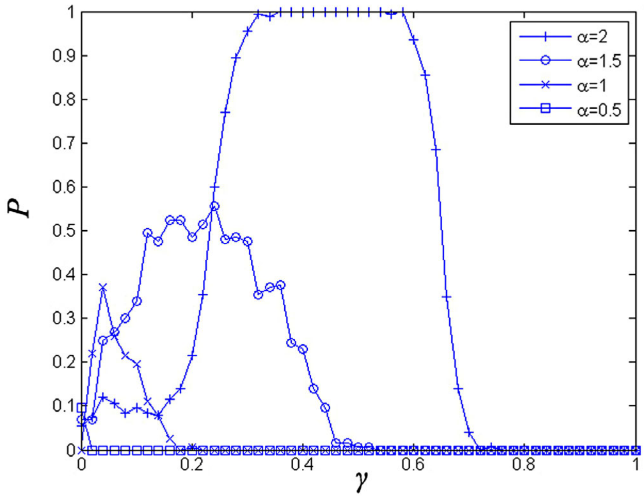

Fig. 3. The success probability P of logic gate OR versus scale γ with different characteristic exponent $\alpha = 2$, 1.5, 1, 0.5, respectively. Here, $\beta = 0$, $\delta = 0$.

For a fixed skewness β, the success probability P as a function of the scale γ is plotted at four different characteristic exponent α (Fig. 3). It shows that the peak of correct probability is lower when the system is driven by heavy tailed noise than it is driven by Gaussian noise at the optimal band of noise density. But the important point is that at weak noise background, the correct probability driven by heavy tailed noise is larger than which is driven by Gaussian noise. The peak of success probability shifts to left as the characteristic exponent gets smaller. At the same time, Figure 4 indicates that when the characteristic exponent is too small, i.e. $\alpha < 1$, the phenomenon of LSR disappears regardless of the value of other parameters.

3.2 Effect of skewness β on LSR

We now consider how the skewness β affects the phenomenon of LSR. Except for the characteristic exponent α, skewness β is another important parameter of α-stable distribution. If $\beta < 0$, the distribution is skewed with the left tail of the distribution heavier than the right, and vice versa. If $\beta = 0$, the distribution is symmetric (Fig. 1b).

Figure 5 shows the responses of the system (1) subjected to Gaussian white noise and α-stable noise with different skewness β when the characteristic exponent $\alpha = 1.5$. It indicates that LSR phenomenon can be induced by asymmetric α-stable noise and the results is even better than symmetric distribution noise.

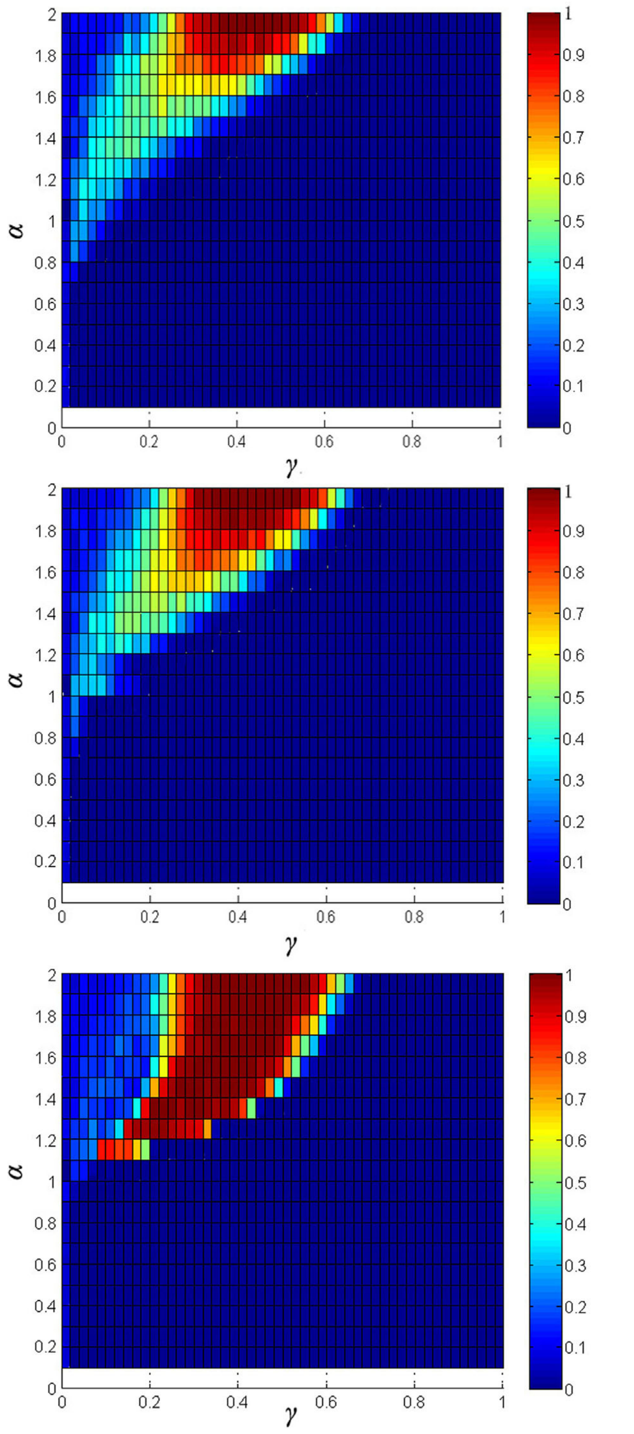

Fig. 4. The success probability P of logic operation OR versus scale γ and characteristic exponent α with different skewness $\beta = 0$, 0.5, 1, respectively (from top to bottom).

Figure 6 demonstrates in relatively wide window of moderate noise, the system yields logic operation with near center probability, i.e. $P \sim 1$, when $|\beta| = 1$. We also note there are two peaks of success probability when $\alpha = 1.5$, $|\beta| = 1$, since the first peak is too low (<0.5), its effects on the output is ignored. The important point is that when the characteristic exponent α is big, i.e. $\alpha > 1.8$, the skewness β has less effect on the results of responses. And in the small α regime, the skewness β has stronger effect on the output of the system (Fig. 7). This is because change in β induces only minor changes in the shape of α-stable density when α is large. Even though, we stress that no matter what the characteristic exponent α values, the asymmetric stable distribution noise is better to stimulate LSR phenomenon than the symmetric noise. Also, we find that which side the distribution lies also affects the result of LSR. More specifically, the plateau of the moderate noise is wider and the peak correct probability is higher at $\beta = 1$, when the nonlinear system obtains OR logic. On the other hand, the behavior of LSR phenomenon to get AND logic performs better when $\beta = -1$.

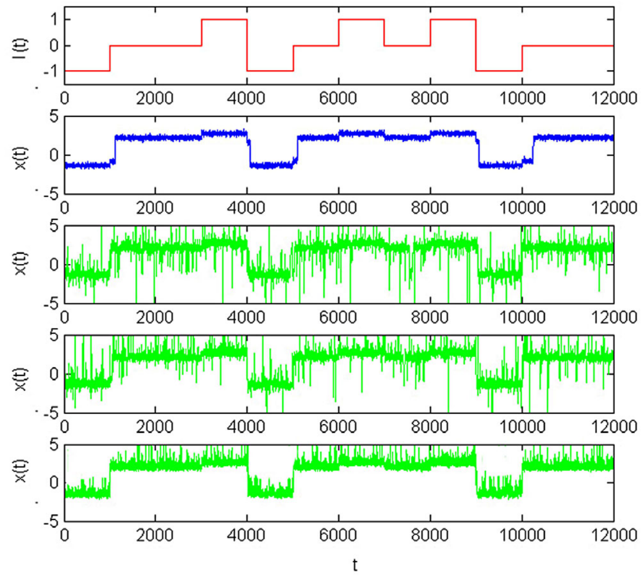

Fig. 5. Panels from top to bottom, the input signal $I(t)$, response of the system to Gaussian noise, responses to α-stable distribution noise with $\beta = 0$, 0.5, 1, respectively. Here, $\alpha = 1.5$, the intensity of Gaussian noise $D = 0.8$, $\gamma = D/\sqrt{2}$.

3.3 Discussion

In this section, we will further analyze the characteristic of LSR driven by α-stable noise. Figure 8 depicts the length of noise scale γ to get correct logical output versus characteristic exponent α and skewness β. It shows that when $\alpha < 1$, it is not possible to get LSR phenomenon. This can be explained by fractional Fokker-Planck equations (FFPEs). For the system (1), when $\beta = \delta = 0$, the statistically equivalent description for the dynamical probability density function (PDF) $\rho(x,t)$, is governed by the following FFPE:

$$\frac{\partial \rho(x,t)}{\partial t} = -\frac{\partial}{\partial x} F(x,t)\,\rho(x,t) + \gamma \frac{\partial^{\partial}}{\partial |x|^{\alpha}} \rho(x,t), \qquad (5)$$

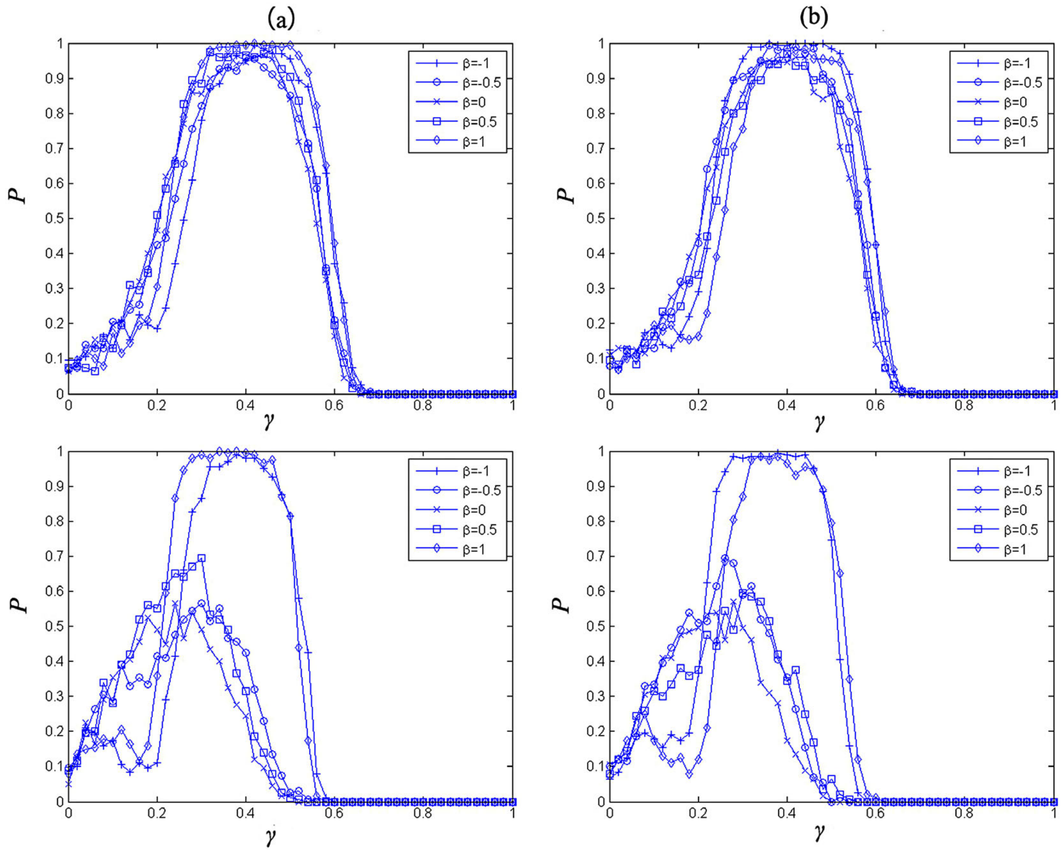

Fig. 6. In the OR (a) and AND (b) gate, the success probability P versus scale γ is shown with the characteristic exponent $\alpha = 1.8$ and $\alpha = 1.5$ (from top to bottom), with $\beta = -1$, –0.5, 0, 0.5, 1, respectively.

where the Riesz space fractional derivative term can be defined through the Liouville-Weyl derivative as [27]:

$$\frac{\partial^\alpha \rho(x,t)}{\partial |x|^\alpha} = \frac{-1}{2\cos(\pi\alpha/2)\,\Gamma(2-\alpha)} \times \frac{\partial^2}{\partial x^2} \int_{-\infty}^{\infty} \frac{\rho(x',t)}{|x-x'|^{\alpha-1}} dx', \quad (6)$$

where Γ denotes the Gamma function. It can be inferred from (5) and (6), when $\alpha < 1$, the effective potential system is no longer bistable. So no matter how the other parameters value, LSR behavior cannot be obtained. At the same time, when $\alpha > 1.8$, skewness has little effects on the correct probability. When α values between 1.8 and 1, the influence of skewness is distinct (Fig. 8). Asymmetric noise is experimented to be better to get the reliable logical output. And the higher the absolute value of skewness gets the more reliable the logical output becomes. Figure 8 shows when the logical operation is OR, the right skewing distribution noise is better to drive the system to obtain LSR phenomenon, vice versa. The result is the same as analyzing the skewness parameter alone.

4 Conclusion

In this paper, LSR phenomenon in a bistable system driven by α-stable noise is investigated. α-stable distribution includes many well-known distributions such as the normal distribution and the Cauchy distribution as special cases. We present that α-stable noise can be used to induce LSR, which display richer dynamical behaviors than systems driven by only white Gaussian noises. Four major results are presented. First of all, with fixed skewness parameter, the peak of correct probability to get right logical output decreases as the characteristic exponent gets smaller. But at the same time, heavy tailed noise can get better logical outcome at a smaller noise level. Secondly, with fixed characteristic exponent, asymmetric distribution noise can realize higher correct output probability than symmetric distribution noise. Although the effect of skewness is not obvious when α is bigger than 1.8, it becomes significant when $\alpha < 1.8$. Thirdly, which side the tail skews affects the reliable of logical outcome too, and the distribution with the left tail heavier than right behaves better to get AND logic operation, vice versa. At last, FFPE is displayed to show that when $\alpha < 1$, no matter how other parameters value, LSR behavior will not be obtained driven by α-stable noise.

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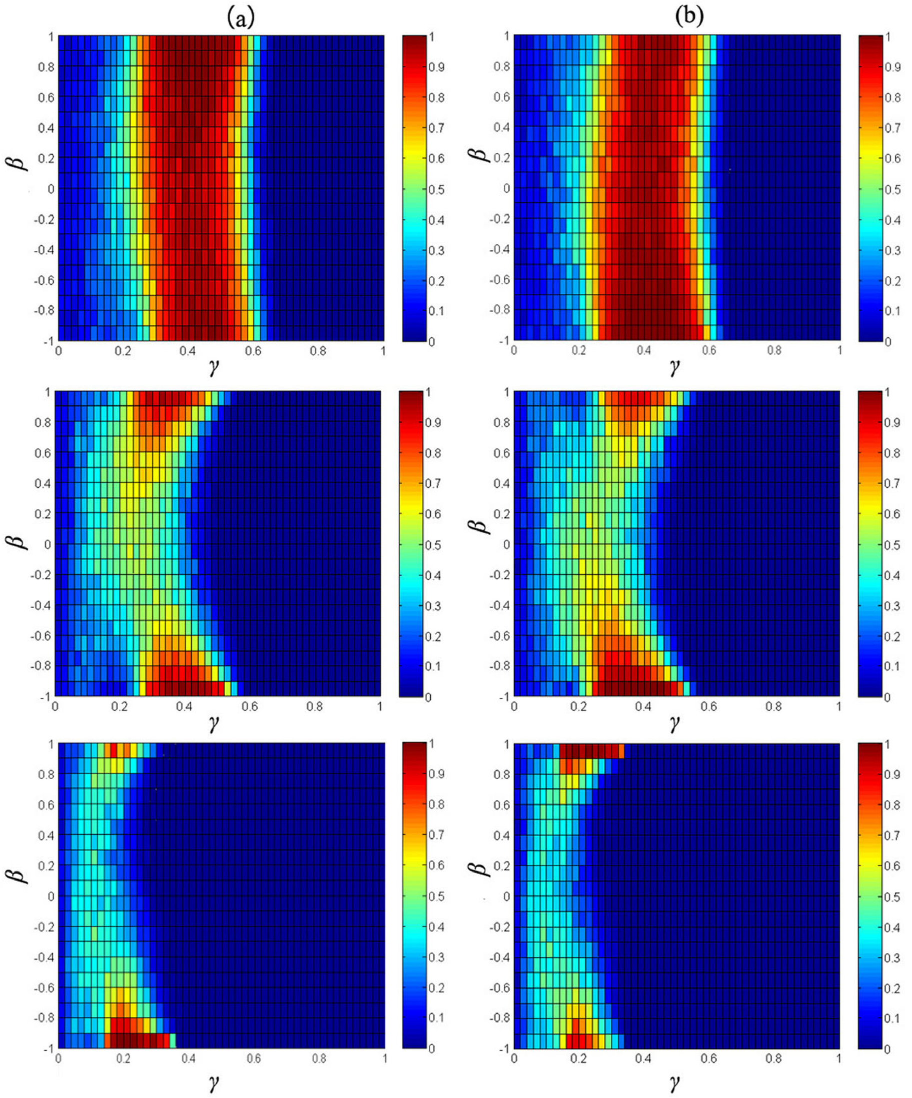

Fig. 7. Density map of P for the OR (a) and AND (b) logic operation versus noise scale γ (x axis) and skewness β (y axis) is shown with the characteristic exponent $\alpha = 1.8$, 1.5, 1.2 (from top to bottom).

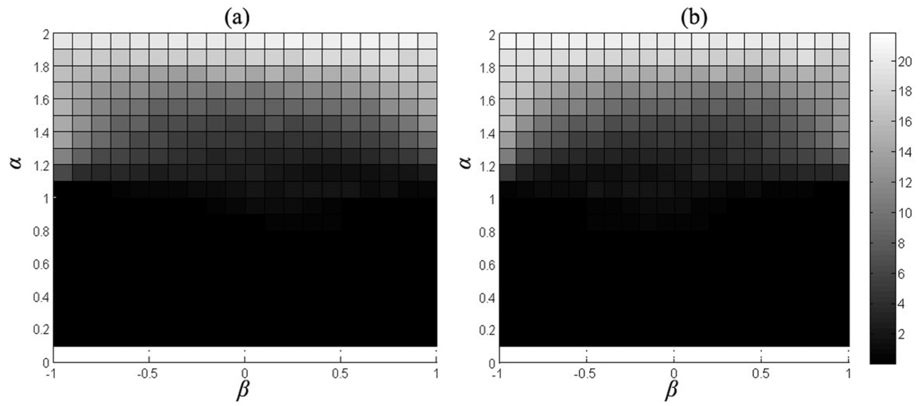

Fig. 8. In the OR (left) and AND (right) gate, the length of noise scale γ to get correct logical output versus characteristic exponent α (y axis) and skewness β (x axis).

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