Slides for Fuzzy Sets, Ch. 2 of Neuro-Fuzzy and Soft Computing

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Fuzzy Sets: Outline

Introduction

Basic definitions and terminology

Set-theoretic operations

MF formulation and parameterization

- MFs of one and two dimensions
- Derivatives of parameterized MFs

More on fuzzy union, intersection, and complement

- Fuzzy complement
- Fuzzy intersection and union
- Parameterized T-norm and T-conorm

A Case for Fuzzy Logic

"So far as the laws of mathematics refer to reality, they are not certain. And so far as they are certain, they do not refer to reality."

- Albert Einstein

Probability versus Fuzziness

I am thinking of a random shape (circle, square, or triangle). What is the probability that I am thinking of a circle?

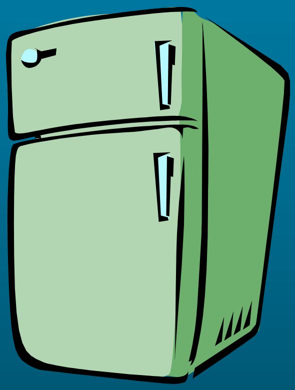
Which statement is more accurate?

- It is probably a circle.
- It is a fuzzy circle.

Probability versus Fuzziness

Two similar but different situations:

- There is a 50% chance that there is an apple in the fridge.
- There is half of an apple in the fridge.



Paradoxes

A heterological word is one that does not describe itself. For example, "long" is heterological, and "monosyllabic" is heterological.

Is "heterological" heterological?

Paradoxes

Bertrand Russell's barber paradox (1901)

The barber shaves a man if and only if he does not shave himself. Who shaves the barber? …

S: The barber shaves himself

Use t(S) to denote the truth of S

S implies not-S, and not-S implies S

Therefore, t(S) = t(not-S) = 1 – t(S)

t(S) = 0.5

Similarly, "heterological" is 50% heterological

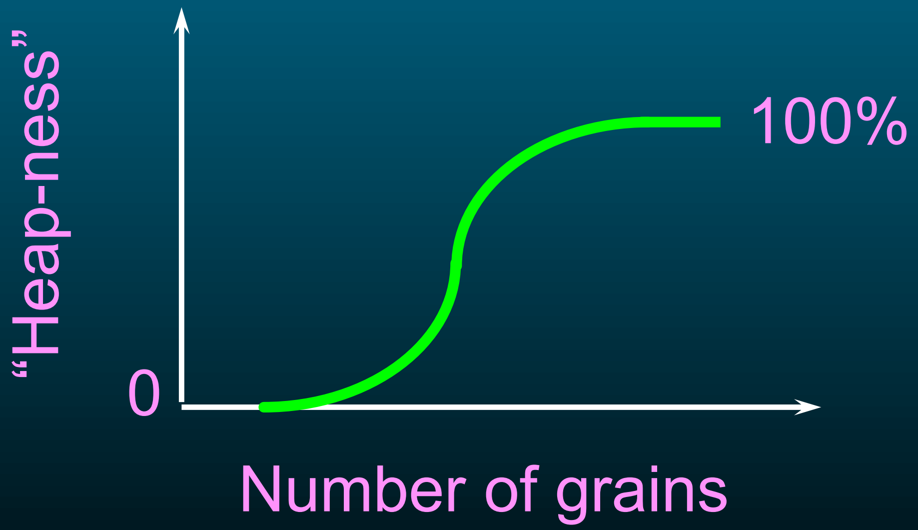
Paradoxes

Sorites paradox:

Premise 1: One million grains of sand is a heap

Premise 2: A heap minus one grain is a heap

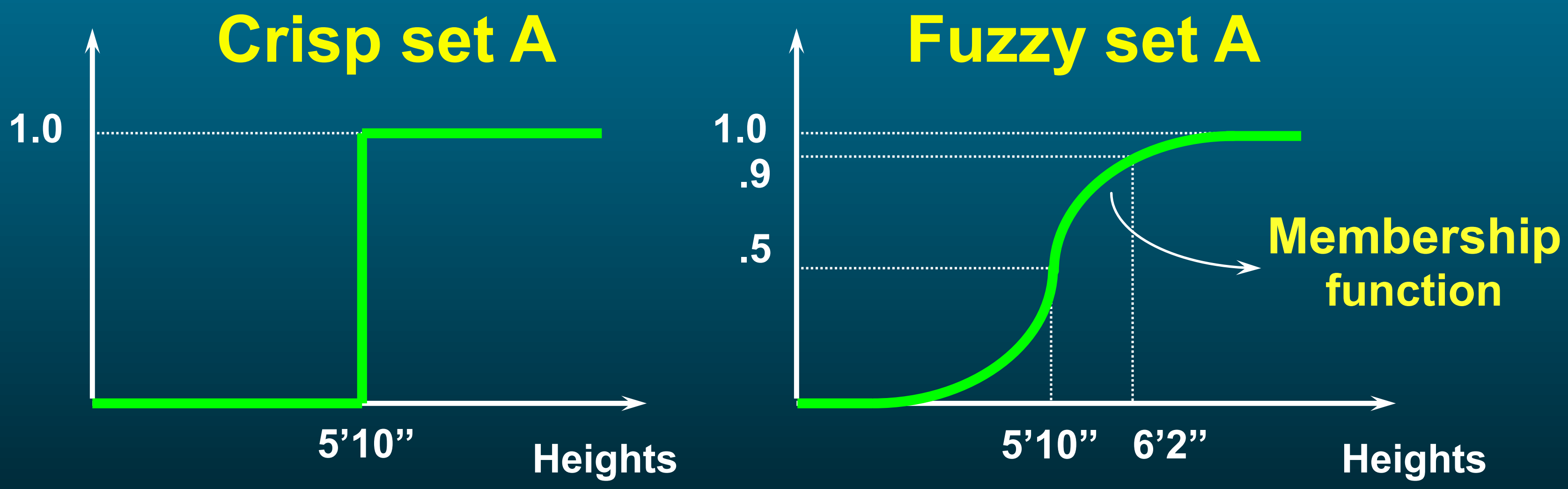
Question: Is one grain of sand a heap?



Fuzzy Sets

Sets with fuzzy boundaries

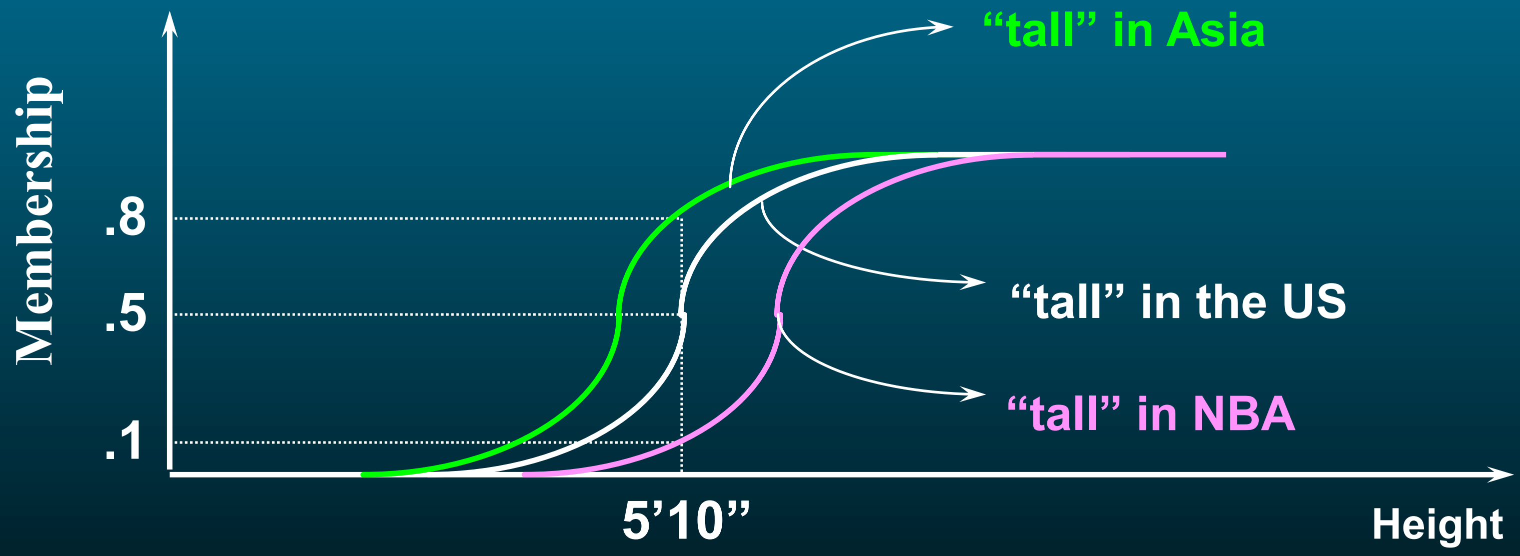
A = Set of tall people



Membership Functions (MFs)

Characteristics of MFs:

- Subjective measures
- Not probability functions



Fuzzy Sets

Formal definition:

A fuzzy set *A* in *X* is expressed as a set of ordered pairs:

$$A = \{(x, \mu_A(x)) | x \in X\}$$

Fuzzy set — A

Membership function (MF) — $\mu_A(x)$

Universe or universe of discourse — X

A fuzzy set is totally characterized by a membership function (MF).

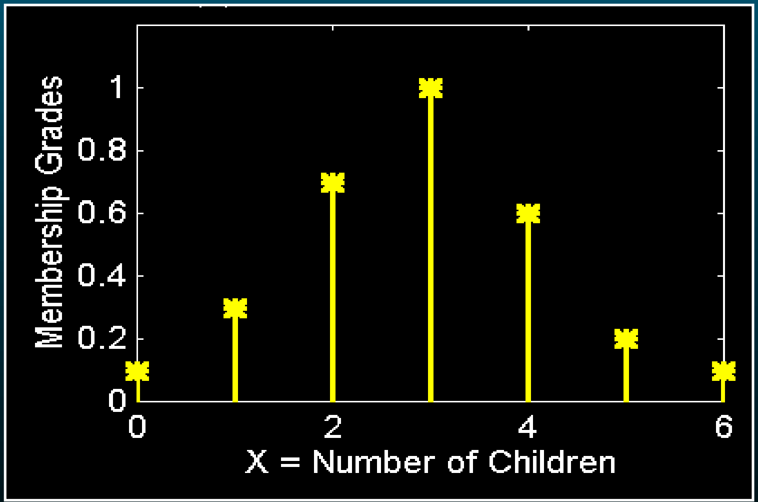
Fuzzy Sets with Discrete Universes

Fuzzy set C = "desirable city to live in"

- X = {SF, Boston, LA} (discrete and nonordered)
- C = {(SF, 0.9), (Boston, 0.8), (LA, 0.6)}

Fuzzy set A = "sensible number of children"

- X = {0, 1, 2, 3, 4, 5, 6} (discrete universe)
- A = {(0, .1), (1, .3), (2, .7), (3, 1), (4, .6), (5, .2), (6, .1)}

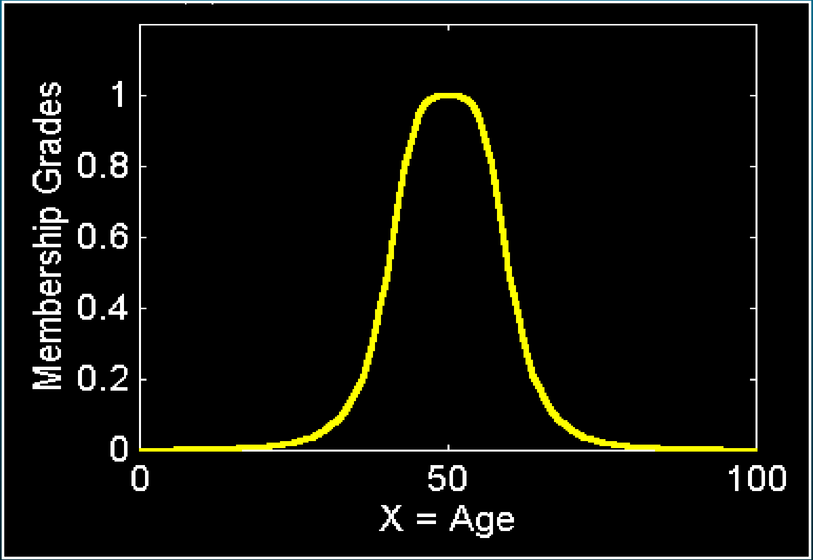


Fuzzy Sets with Cont. Universes

Fuzzy set B = "about 50 years old"

- X = Set of positive real numbers (continuous)
- B = {(x, μ_B(x)) | x in X}

$$\mu_B(x) = \frac{1}{1 + \left(\frac{x - 50}{10}\right)^2}$$



Alternative Notation

A fuzzy set A can be alternatively denoted as follows:

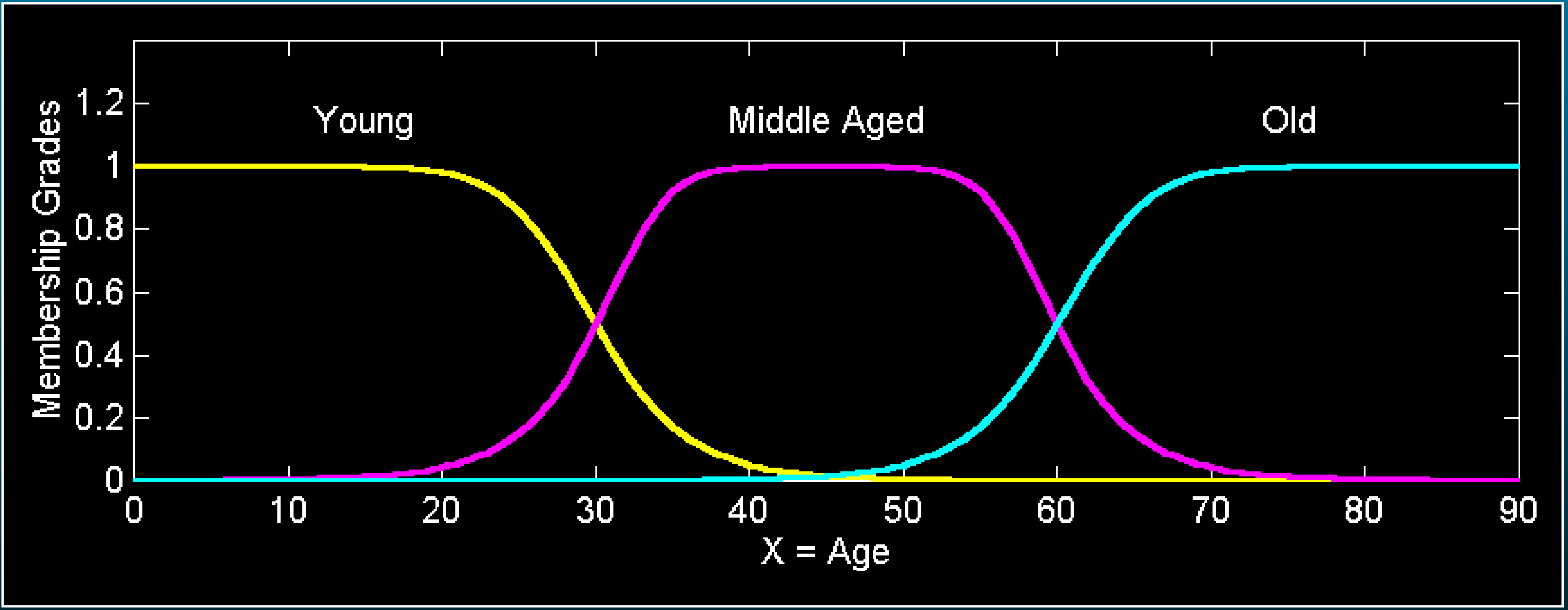
X is discrete → $$A = \sum_{x_i \in X} \mu_A(x_i) / x_i$$

X is continuous → $$A = \int_X \mu_A(x) / x$$

Note that Σ and integral signs stand for the union of membership grades; "/" stands for a marker and does not imply division.

Fuzzy Partition

Fuzzy partitions formed by the linguistic values "young", "middle aged", and "old":



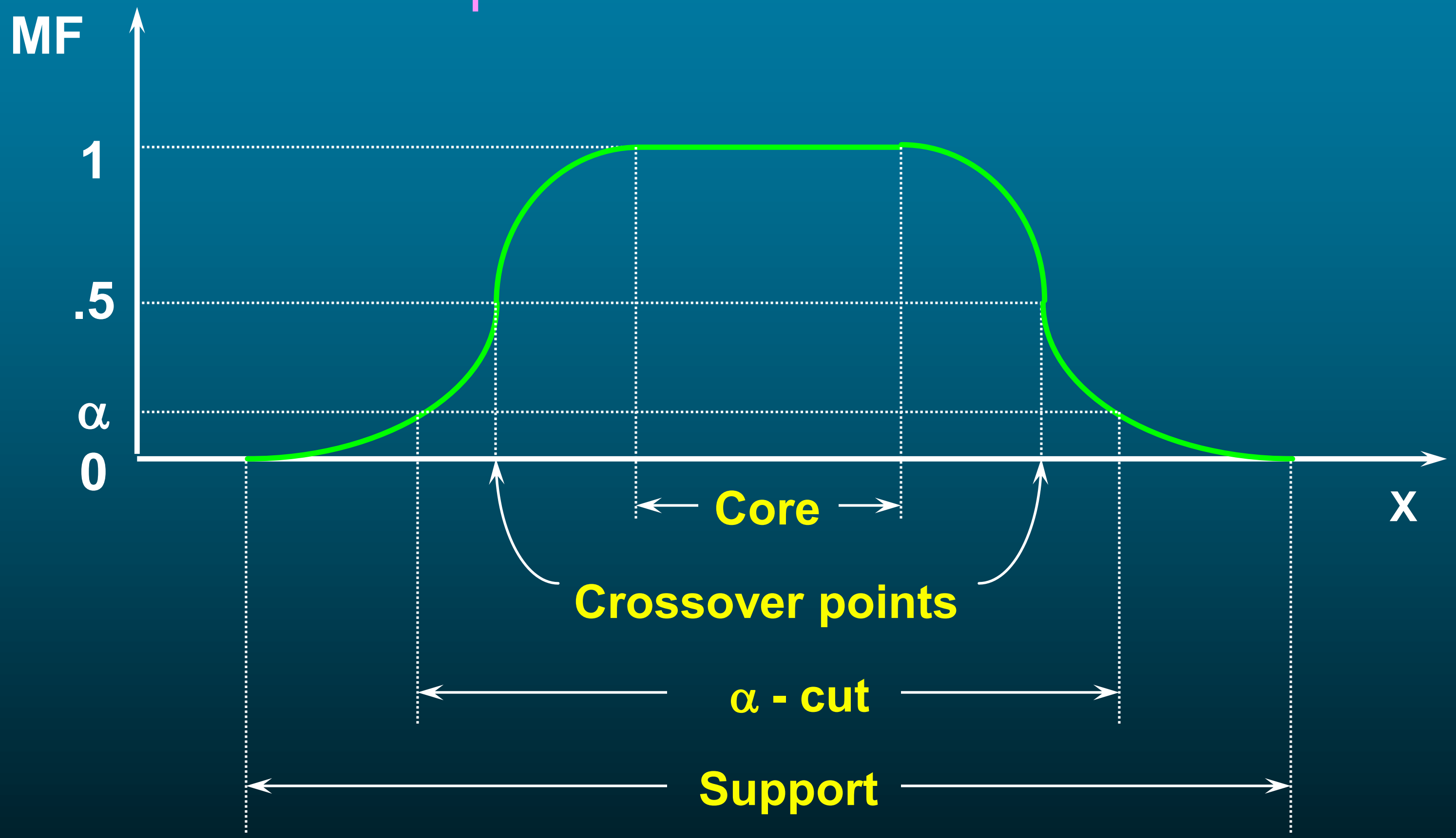
lingmf.m

More Definitions

- Support
- Core
- Normality
- Crossover points
- Fuzzy singleton
- α-cut, strong α-cut
- Convexity
- Fuzzy numbers
- Bandwidth
- Symmetricity
- Open left or right, closed

MF Terminology

These expressions are all defined in terms of *x*.

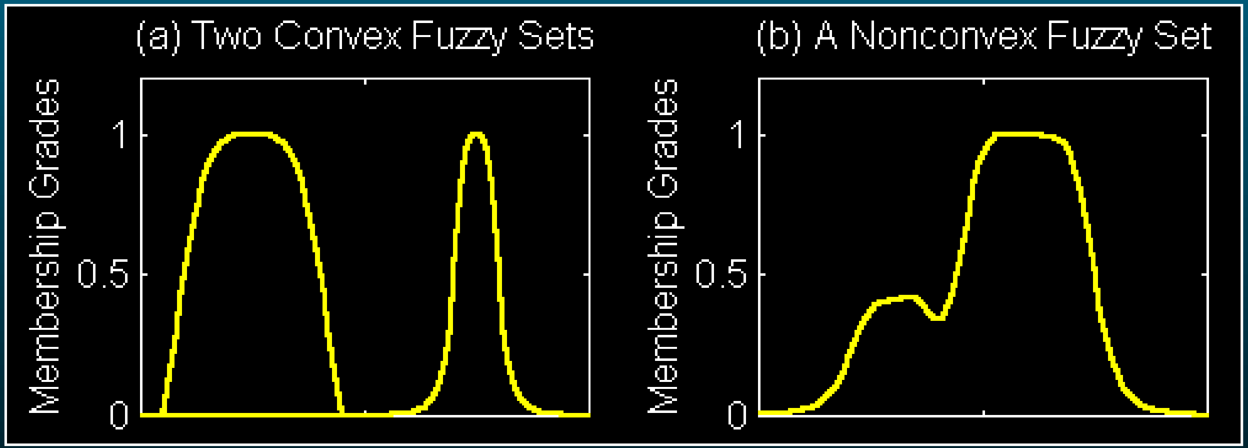


Convexity of Fuzzy Sets

A fuzzy set *A* is convex if for any λ in [0, 1],

$$\mu_A(\lambda x_1 + (1-\lambda)x_2) \geq \min(\mu_A(x_1), \mu_A(x_2))$$

A is convex if all its α-cuts are convex. (How do you measure the convexity of an α-cut?)



convexmf.m

Set-Theoretic Operations

Subset:

$$A \subseteq B \Leftrightarrow \mu_A \le \mu_B \text{ for all } x$$

Complement:

$$\bar{A} = X - A \Leftrightarrow \mu_{\bar{A}}(x) = 1 - \mu_A(x)$$

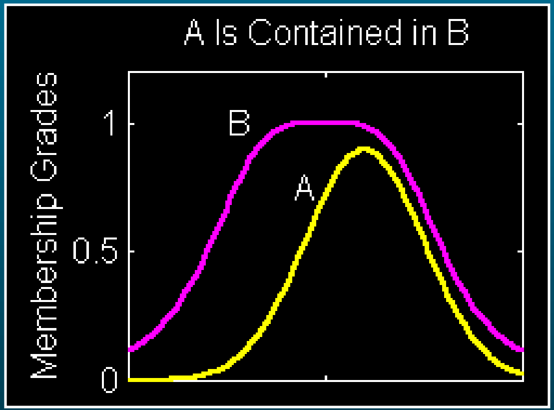
Union:

$$C = A \cup B \Leftrightarrow \mu_c(x) = \max(\mu_A(x), \mu_B(x)) = \mu_A(x) \vee \mu_B(x)$$

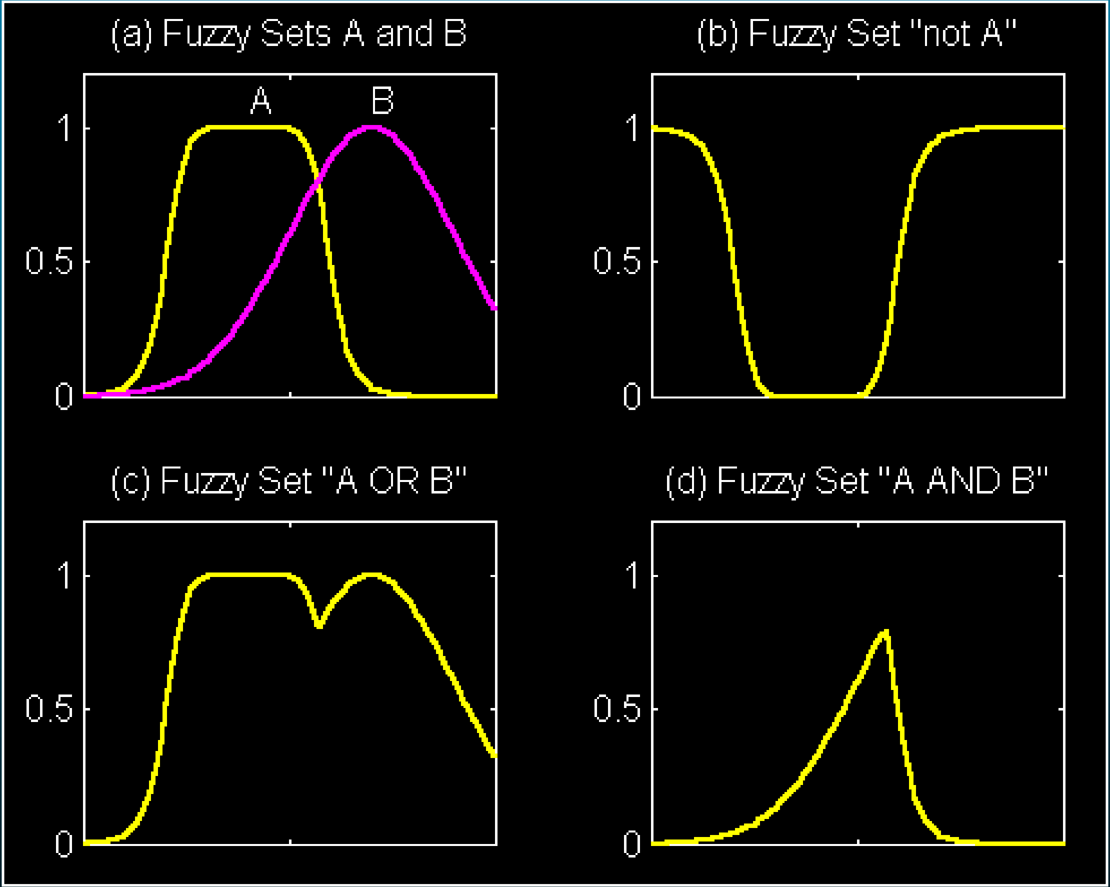
Intersection:

$$C = A \cap B \Leftrightarrow \mu_c(x) = \min(\mu_A(x), \mu_B(x)) = \mu_A(x) \wedge \mu_B(x)$$

Set-Theoretic Operations



subset.m



fuzsetop.m

MF Formulation

Triangular MF:

$$\text{trimf}(x;a,b,c) = \max\left(\min\left(\frac{x-a}{b-a}, \frac{c-x}{c-b}\right), 0\right)$$

Trapezoidal MF:

$$\text{trapmf}(x;a,b,c,d) = \max\left(\min\left(\frac{x-a}{b-a}, 1, \frac{d-x}{d-c}\right), 0\right)$$

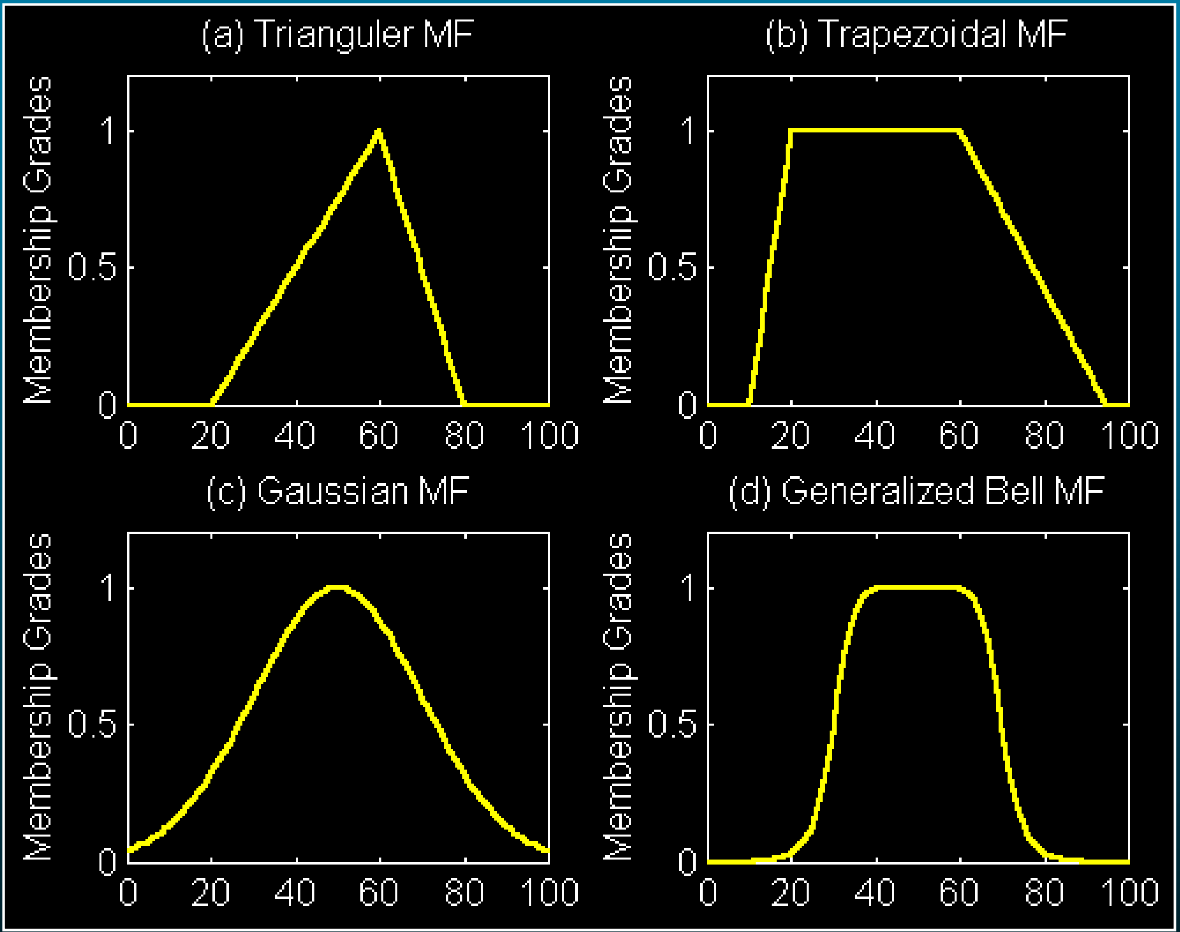
Gaussian MF:

$$\text{gaussmf}(x;c,\sigma) = e^{-\frac{1}{2}\left(\frac{x-c}{\sigma}\right)^2}$$

Generalized bell MF:

$$\text{gbellmf}(x;a,b,c) = \frac{1}{1+\left|\frac{x-c}{a}\right|^{2b}}$$

MF Formulation



disp_mf.m

MF Formulation

Sigmoidal MF:

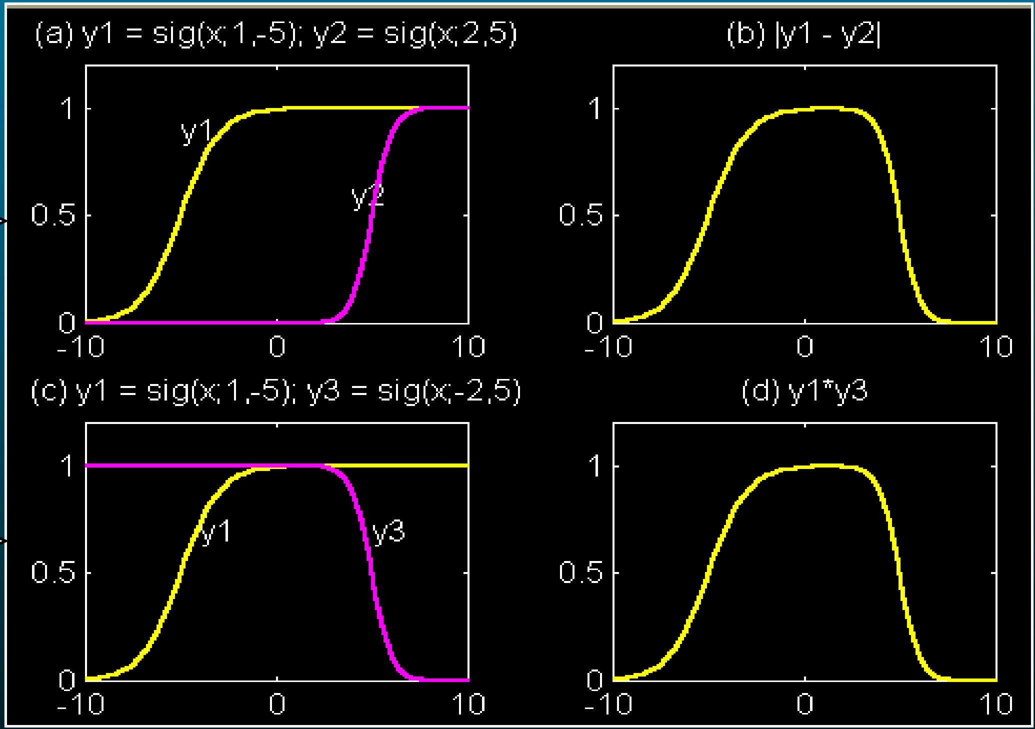
$$\text{sigmf}(x; a, c) = \frac{1}{1 + e^{-a(x-c)}}$$

c = crossover point
a controls the slope, and right/left

Extensions:

Abs. difference of two sig. MF (open right MFs)

Product of two sig. MFs



disp_sig.m

MF Formulation

L-R (left-right) MF:

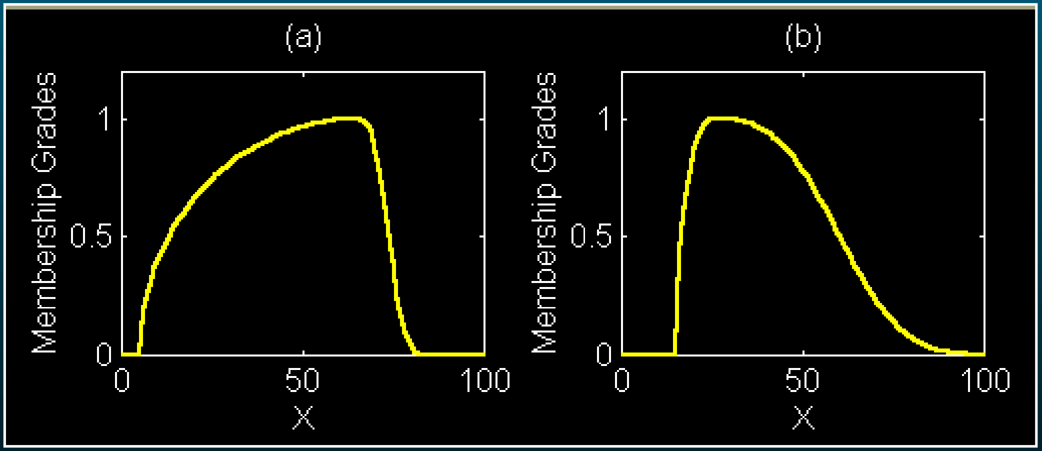
$$\mathrm{LR}(x;c,a,b)=\begin{cases}F_L\left(\frac{c-x}{a}\right), & x<c\\ F_R\left(\frac{x-c}{b}\right), & x\geq c\end{cases}$$

Example:

$$F_L(x)=\sqrt{\max(0,1-x^2)}$$

$$F_R(x)=\exp(-|x|^3)$$

c=65
a=60
b=10

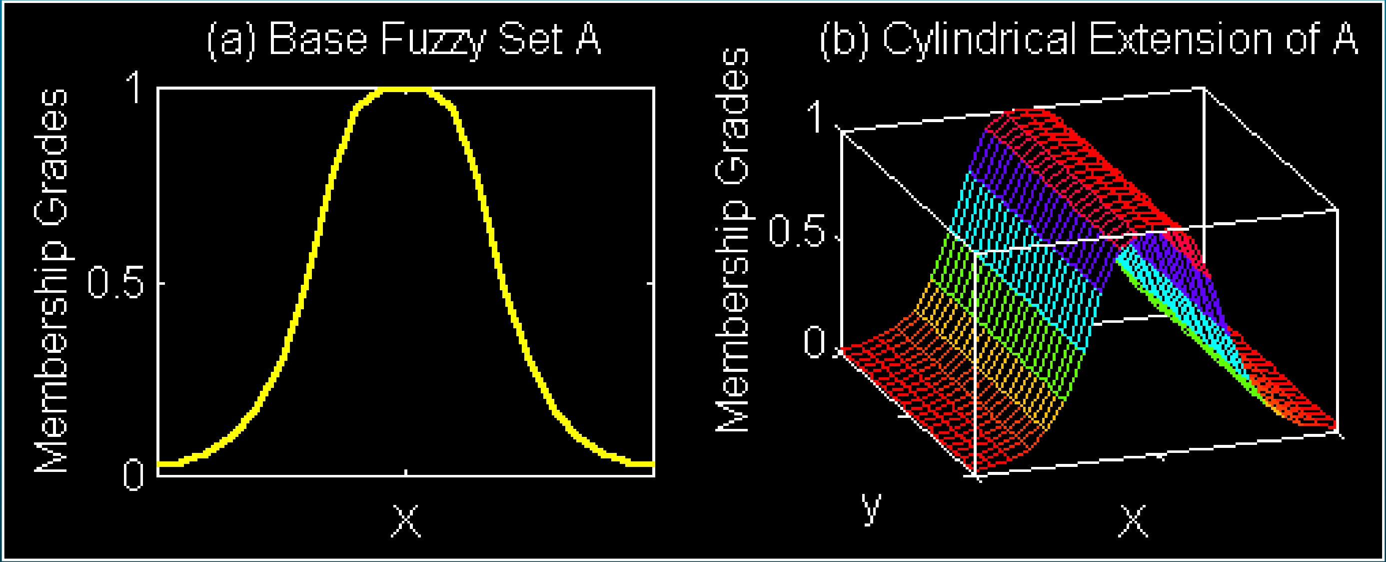


c=25
a=10
b=40

difflr.m

Cylindrical Extension

Base set A **Cylindrical Ext. of A**

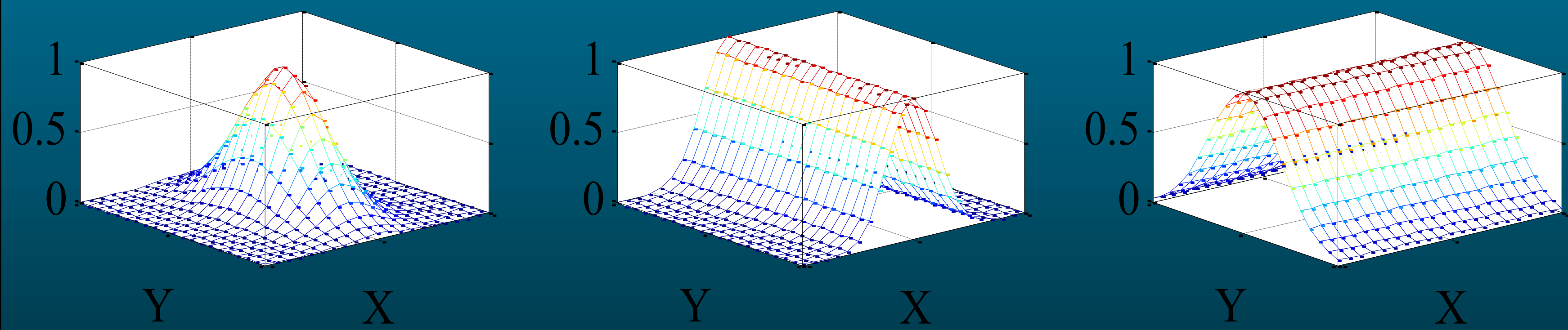


cyl_ext.m

2D MF Projection

Two-dimensional MF | **Projection onto X** | **Projection onto Y**

(a) A Two-dimensional MF | (b) Projection onto X | (c) Projection onto Y



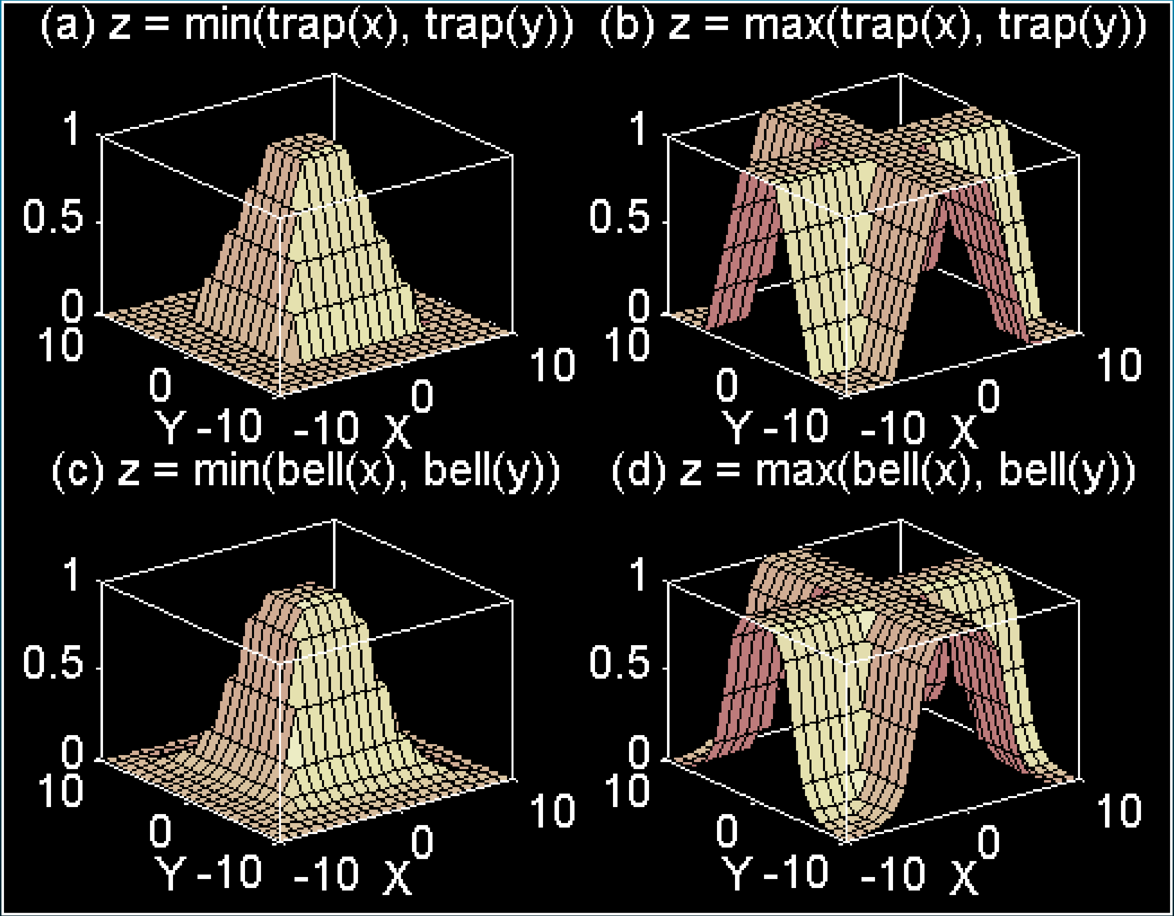
$$\mu_R(x, y)$$

$$\mu_A(x) = \max_y \mu_R(x, y)$$

$$\mu_B(y) = \max_x \mu_R(x, y)$$

project.m

2D Membership Functions



2dmf.m

Fuzzy Complement *N*(*a*) : [0,1]→[0,1]

General requirements of fuzzy complement:

- Boundary: N(0)=1 and N(1) = 0
- Monotonicity: N(a) > N(b) if a < b
- Involution: N(N(a)) = a

Two types of fuzzy complements:

- Sugeno's complement (Michio Sugeno):

$$N_s(a) = \frac{1-a}{1+sa}$$

- Yager's complement (Ron Yager, Iona College):

$$N_w(a) = (1-a^w)^{1/w}$$

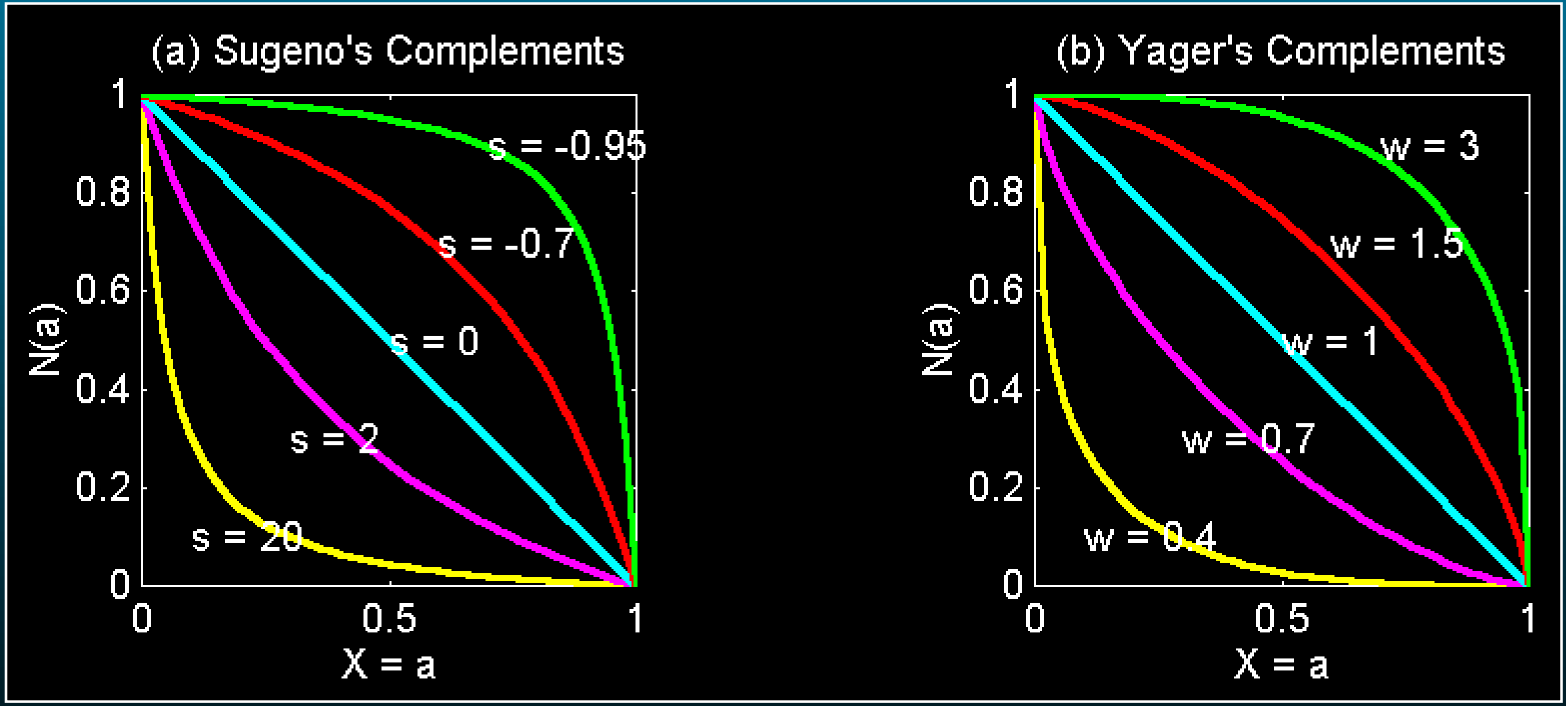
Fuzzy Complement

Sugeno's complement:

$$N_s(a) = \frac{1-a}{1+sa}$$

Yager's complement:

$$N_w(a) = (1-a^w)^{1/w}$$



negation.m

Fuzzy Intersection: T-norm

Analogous to AND, and INTERSECTION

Basic requirements:

- Boundary: $T(0, 0) = 0$, $T(a, 1) = T(1, a) = a$
- Monotonicity: $T(a, b) \leq T(c, d)$ if $a \leq c$ and $b \leq d$
- Commutativity: $T(a, b) = T(b, a)$
- Associativity: $T(a, T(b, c)) = T(T(a, b), c)$

Four examples (page 37):

- Minimum: $T_m(a, b)$
- Algebraic product: $T_a(a, b)$
- Bounded product: $T_b(a, b)$
- Drastic product: $T_d(a, b)$

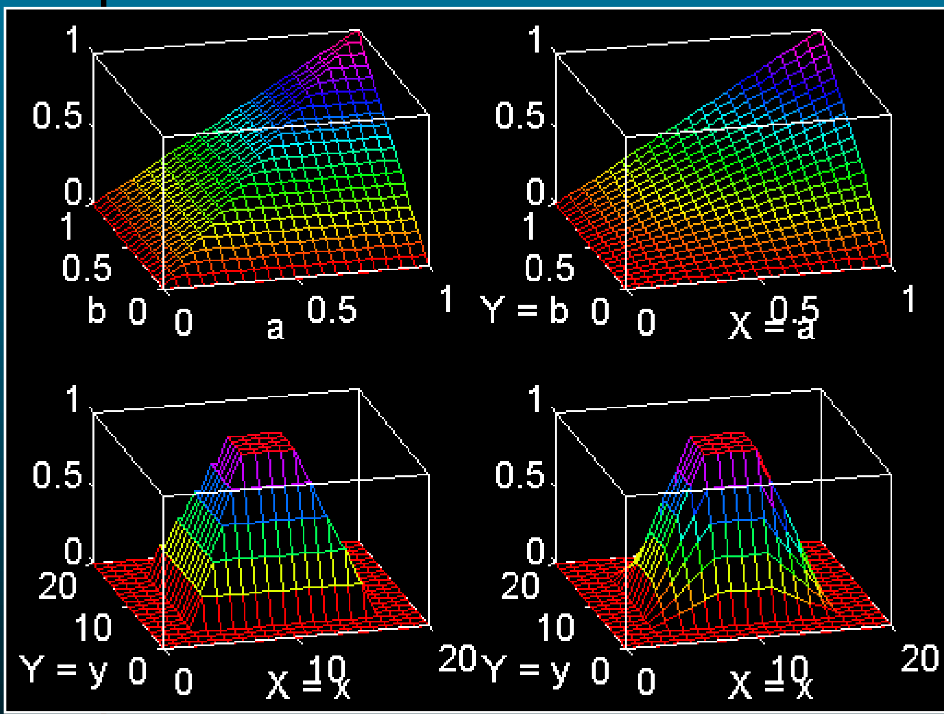
T-norm Operator

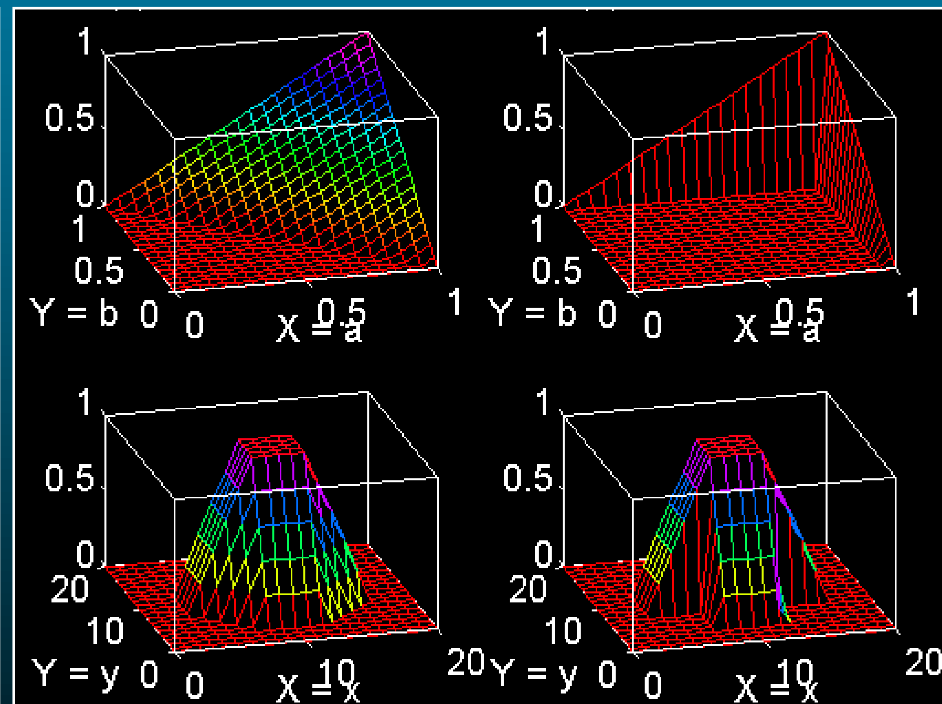
Minimum: $T_m(a, b)$

Algebraic product: $T_a(a, b)$

Bounded product: $T_b(a, b)$

Drastic product: $T_d(a, b)$





tnorm.m

Fuzzy Union: T-conorm or S-norm

Analogous to OR, and UNION

Basic requirements:

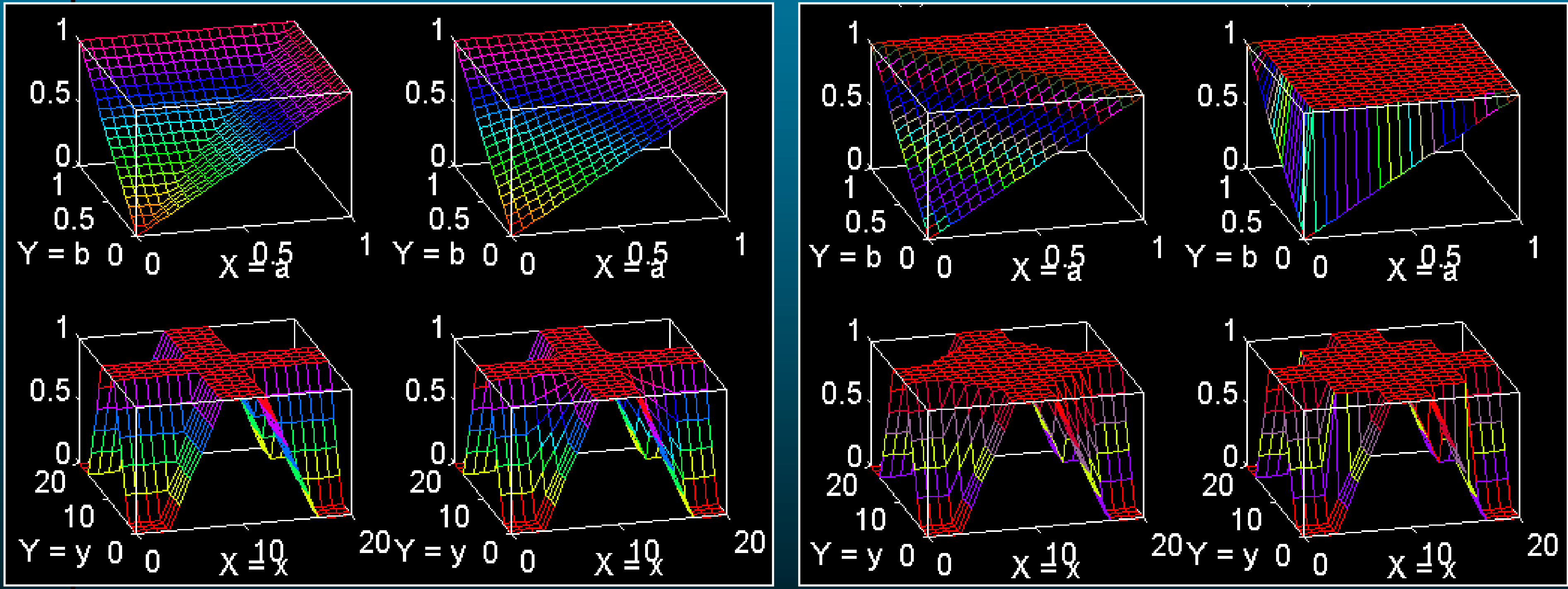
- Boundary: $S(1, 1) = 1$, $S(a, 0) = S(0, a) = a$
- Monotonicity: $S(a, b) \leq S(c, d)$ if $a \leq c$ and $b \leq d$
- Commutativity: $S(a, b) = S(b, a)$
- Associativity: $S(a, S(b, c)) = S(S(a, b), c)$

Four examples (page 38):

- Maximum: $S_m(a, b)$
- Algebraic sum: $S_a(a, b)$
- Bounded sum: $S_b(a, b)$
- Drastic sum: $S_d(a, b)$

T-conorm or S-norm

Maximum: $S_m(a, b)$	Algebraic sum: $S_a(a, b)$	Bounded sum: $S_b(a, b)$	Drastic sum: $S_d(a, b)$



tconorm.m

Generalized DeMorgan's Law

T-norms and T-conorms are duals which support the generalization of DeMorgan's law:

- T(a, b) = N(S(N(a), N(b))): a and b = not(not a, or not b)
- S(a, b) = N(T(N(a), N(b))): a or b = not(not a, and not b)

$T_m(a, b)$	$S_m(a, b)$
$T_a(a, b)$	$S_a(a, b)$
$T_b(a, b)$	$S_b(a, b)$
$T_d(a, b)$	$S_d(a, b)$

Parameterized T-norm and S-norm

Parameterized T-norms and dual T-conorms have been proposed by several researchers:

- Yager
- Schweizer and Sklar
- Dubois and Prade
- Hamacher
- Frank
- Sugeno
- Dombi