Fuzzy Rules and Fuzzy Reasoning

(chapter 3)

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(adapted from slides by R. Jang)

Fuzzy Reasoning: The Big Picture

Useful for:
- Automatic Control
- Expert Systems
- Pattern Recognition
- Time Series Prediction
- Data Classification

There are different schemes to accomplish these goals depending on our understanding and boundary conditions of the world.

Extending:
- crisp domains to fuzzy domains: Extension Principle
- n-ary fuzzy relations: Fuzzy Relations
- fuzzy domains to fuzzy domains: Fuzzy Inference (fuzzy rules, compositional rules of inference)

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Outline

- Extension principle
- Fuzzy relations
- Fuzzy IF-THEN rules
- Compositional rule of inference
- Fuzzy reasoning

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Extension Principle

Extends crisp domains of mathematical expressions to fuzzy domains

A is a fuzzy set on *X* :

$$A = \mu_A(x_1)/x_1 + \mu_A(x_2)/x_2 + \cdots + \mu_A(x_n)/x_n$$

The image of *A* under *f( )* is a fuzzy set *B:,* i.e., *B=f(A)*

$$B = \mu_B(x_1)/y_1 + \mu_B(x_2)/y_2 + \cdots + \mu_B(x_n)/y_n$$

where $y_i = f(x_i)$, $i = 1$ to n.

If *f( )* is a many-to-one mapping, then

$$\mu_B(y) = \max_{x=f^{-1}(y)} \mu_A(x)$$

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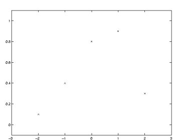
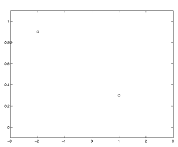
Extension Principle: Example

A=0.1/-2+0.4/-1+0.8/0+0.9/1+0.3/2



$f(x)=x^2-3$

B=0.1/1+0.4/-2+0.8/-3+0.9/-2+0.3/1
=0.8/-3+(0.4v0.9)/-2+(0.1v0.3)/1
=0.8/-3+0.9/-2+0.3/1



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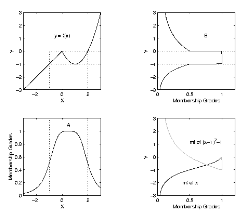
Extension Principle, continuous vars.

Let $\mu_A(x) = bell(x; 1.5, 2, 0.5)$

and $f(x) = \begin{cases} (x-1)^2 - 1 & \text{if } x \geq 0 \\ x & \text{if } x < 0 \end{cases}$



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Fuzzy Relations

A fuzzy relation *R* is a 2D MF:

$$R = \{((x, y), \mu_R(x, y)) | (x, y) \in X \times Y\}$$

Examples:

- **x depends on y (x and y are events)**
- **If x is large, then y is small (x is an observed reading and Y is a corresponding action)**
- **y is much greater than x (x and y are numbers)**

$$\mu_R(x, y) = \begin{cases} \frac{y - x}{x + y + 2} & \text{if } y \ge x \\ 0 & \text{if } y < x \end{cases}$$

If X={3,4,5} and Y={3,4,5,6,7}

- **Express fuzzy relation as a relation matrix**

$$R = \begin{bmatrix} 0 & 0.11 & 0.20 & 0.27 & 0.33 \\ 0 & 0 & 0.09 & 0.17 & 0.23 \\ 0 & 0 & 0 & 0.08 & 0.14 \end{bmatrix}$$

Soft Computing: Fuzzy Rules and Fuzzy Reasoning

Max-Min Composition

The max-min composition of two fuzzy relations R_1 (defined on *X* and *Y*) and R_2 (defined on *Y* and *Z*) is

$$\mu_{R_1 \circ R_2}(x, z) = \vee_y [\mu_{R_1}(x, y) \wedge \mu_{R_2}(y, z)]$$

Note: calculation very similar to matrix multiplication

Properties:

- **Associativity:** $R \circ (S \circ T) = (R \circ S) \circ T$
- **Distributivity over union:** $R \circ (S \cup T) = (R \circ S) \cup (R \circ T)$
- **Week distributivity over intersection:** $R \circ (S \cap T) \subseteq (R \circ S) \cap (R \circ T)$
- **Monotonicity:** $S \subseteq T \Rightarrow (R \circ S) \subseteq (R \circ T)$

Soft Computing: Fuzzy Rules and Fuzzy Reasoning

Max-min Composition: Example

Let

R_1="x is relevant to y"

R_2="y is relevant to z"

where

$$R_1 = \begin{bmatrix} 0.1 & 0.3 & 0.5 & 0.7 \\ 0.4 & 0.2 & 0.8 & 0.9 \\ 0.6 & 0.8 & 0.3 & 0.2 \end{bmatrix} \quad R_2 = \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.3 \\ 0.5 & 0.6 \\ 0.7 & 0.2 \end{bmatrix}$$

X={ 1 2 3}

Y={α β χ δ} ***Z={a b}***

Soft Computing: Fuzzy Rules and Fuzzy Reasoning

Max-min Composition: Example

Calculate:

2 is relevant to a

$$R_1 = \begin{bmatrix} 0.1 & 0.3 & 0.5 & 0.7 \\ 0.4 & 0.2 & 0.8 & 0.9 \\ 0.6 & 0.8 & 0.3 & 0.2 \end{bmatrix} \quad R_2 = \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.3 \\ 0.5 & 0.6 \\ 0.7 & 0.2 \end{bmatrix}$$

X={ 1 2 3}

Y={α β χ δ} ***Z={a b}***

$$\mu_{R_1 \circ R_2}(2, a) = \boxed{}$$

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Max-Star Composition

Max-product composition:

$$\mu_{R_1 \circ R_2}(x, z) = \vee_y [\mu_{R_1}(x, y) \mu_{R_2}(y, z)]$$

In general, we have max-* composition:

$$\mu_{R_1 \circ R_2}(x, z) = \vee_y [\mu_{R_1}(x, y) * \mu_{R_2}(y, z)]$$

where * is a T-norm operator.

Even more generally, we have (S-norm)-(T-norm) compositions

Soft Computing: Fuzzy Rules and Fuzzy Reasoning

Linguistic Variables (knowledge rep.)

Precision vs. significance

A <u>numerical variable</u> can take on numerical *(Age = 65)* or linguistic values *(Age is old)*

A linguistic value is characterized by the variable name (age) a fuzzy set.

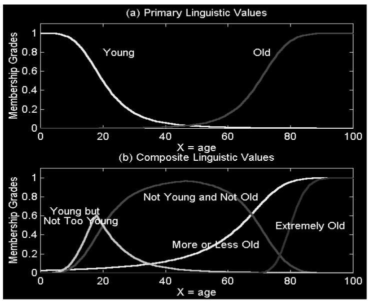
All <u>linguistic values</u> form a <u>term set</u>:

T(age) = {young, not young, very young, ...
middle aged, not middle aged, ...
old, not old, very old, more or less old, ...
not very young and not very old, ...}

The <u>syntactic rule</u> refers to how the linguistic values are generated.

The <u>semantic rule</u> defines the membership value of each linguistic variable.

Linguistic Values (Terms)



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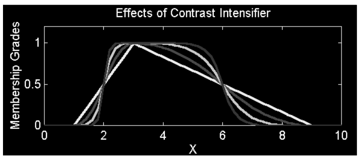
Operations on Linguistic Values

Concentration: ⇨ $CON(A) = A^2$

Dilation: ⇨ $DIL(A) = A^{0.5}$

Contrast Intensification: ⇨ $INT(A) = \begin{cases} 2A^2, & 0 \le \mu_A(x) \le 0.5 \\ \neg 2(\neg A)^2, & 0.5 \le \mu_A(x) \le 1 \end{cases}$



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Fuzzy IF-THEN Rules

General format:

IF x is A THEN y is B

if <antecedent> then <consequent>

Examples:

- **IF pressure is high THEN volume is small.**
- **IF the road is slippery THEN driving is dangerous.**
- **IF a tomato is red THEN it is ripe.**
- **IF the speed is high THEN apply the brake a little.**

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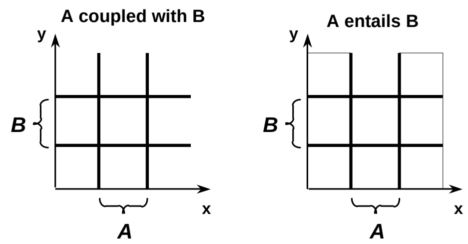
Fuzzy IF-THEN Rules

Two ways to interpret "IF x is A THEN y is B":



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Fuzzy If-Then Rules

Two ways to interpret "If x is A then y is B":

- **A coupled with B: (*A and B*)**

$$R = A \to B = A \times B = \int \mu_A(x) \tilde{*} \mu_B(y) | (x, y)$$

- **A entails B: (*not A or B*)**
  - Material implication $\neg A \cup B$
  - Propositional calculus $\neg A \cup (A \cap B)$
  - Extended propositional calculus $(\neg A \cap \neg B) \cup B$
  - Generalization of modus ponens

$$\mu_R(x, y) = \sup\{c | \mu_A(x) \tilde{*} c \le \mu_B(y) \text{ and } 0 \le c \le 1\}$$

- **Note: these all reduce to *not A or B* in two-valued logic**

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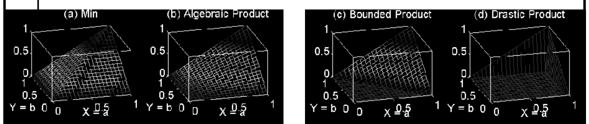
Fuzzy IF-THEN Rules

Fuzzy implication function:

$$\mu_R(x, y) = f(\mu_A(x), \mu_B(y)) = f(a, b)$$

A coupled with B



$\mu_R(x, y) = \min(\mu_A(x), \mu_B(y))$ $\mu_R(x, y) = 0 \vee (\mu_A(x) + \mu_B(y) - 1)$

$\mu_R(x, y) = \mu_A(x)\mu_B(y)$

$$\mu_R(x, y) = \begin{cases} \mu_A(x) \text{ if } \mu_B(y) = 1 \\ \mu_B(y) \text{ if } \mu_A(x) = 1 \\ 0 \text{ otherwise} \end{cases}$$

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Fuzzy IF-THEN Rules

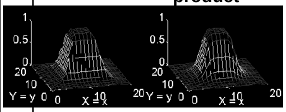
A coupled with B

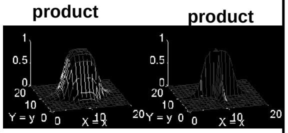
Let

$\mu_A(x) = bell(x;4,3,10)$

$\mu_B(y) = bell(y;4,3,10)$

min | algebraic product | bounded product | drastic product





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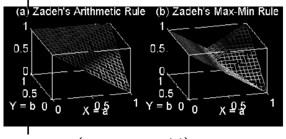
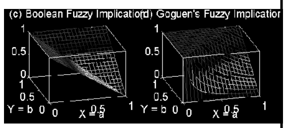
Fuzzy IF-THEN Rules

A entails B

(not an exhaustive list)





$\mu_R(x,y) = \min(1, 1-\mu_A(x)+\mu_B(y))$

$\mu_R(x,y) = \max(\min(\mu_A(x),\mu_B(y)), 1-\mu_A(x))$

$\mu_R(x,y) = \min\left(1, \frac{\mu_A(x)}{\mu_B(y)}\right)$

$\mu_R(x,y) = \max(1-\mu_A(x), \mu_B(y))$

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Soft Computing: Fuzzy Rules and Fuzzy Reasoning

Fuzzy IF-THEN Rules

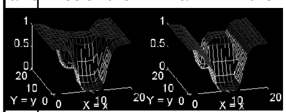
A entails B

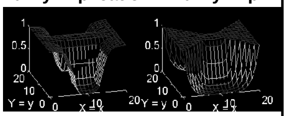
Let

$\mu_A(x) = bell(x;4,3,10)$

$\mu_B(y) = bell(y;4,3,10)$

Zadeh's arithmetic rule | Zadeh's max-min rule | Boolean Fuzzy Implication | Gougen's Fuzzy Impl.





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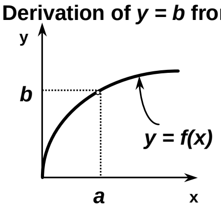
Soft Computing: Fuzzy Rules and Fuzzy Reasoning

Compositional Rule of Inference

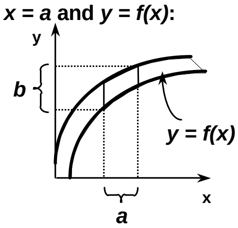
Same idea as max-min composition

Derivation of *y = b* from *x = a* and *y = f(x)*:



***a* and *b*: points**

***y = f(x)* : a curve**



***a* and *b*: intervals**

***y = f(x)* : an interval-valued function**

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Soft Computing: Fuzzy Rules and Fuzzy Reasoning

Compositional Rule of Inference, contd.

To find the resulting interval y=b
(which corresponds to x=a)

- construct a cylindrical extension of a
- find intersection with curve
- project intersection to y-axis

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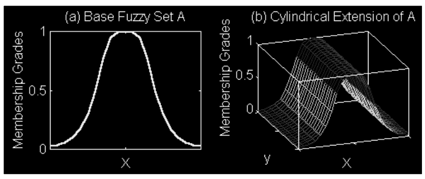
Soft Computing: Fuzzy Rules and Fuzzy Reasoning

Recall: Cylindrical Extension

Base set A | Cylindrical Ext. of A



$\mu_{c(A)}(x,y) = \mu_A(x)$

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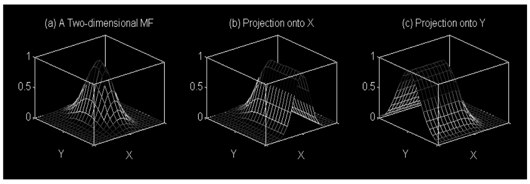
Recall: 2D MF Projection

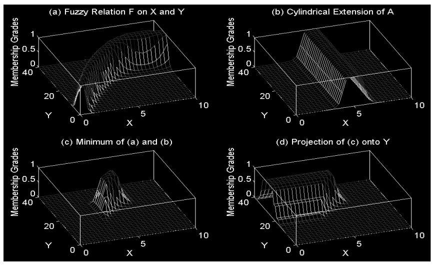
Two-dimensional MF	Projection onto X	Projection onto Y



$\mu_R(x, y)$	$\mu_A(x) = \max_y \mu_R(x, y)$	$\mu_B(y) = \max_x \mu_R(x, y)$

Compositional Rule of Inference

a **is a fuzzy set and** ***y = f(x)*** **is a fuzzy relation:**



Putting it together: Comp. rule of Inf.

Cylindrical extension with base A

$$\mu_{c(A)}(x, y) = \mu_A(x)$$

Intersection of c(A) with F

$$\mu_{c(A)\cap F}(x, y) = \min\left[\mu_{c(A)}(x, y), \mu_F(x, y)\right]$$
$$= \min\left[\mu_A(x), \mu_F(x, y)\right]$$

Projection onto y-axis

$$\mu_B(y) = \max_x \min\left[\mu_A(x), \mu_F(x, y)\right]$$
$$= \vee_x \left[\mu_A(x) \wedge \mu_F(x, y)\right]$$

Representation: $B = A \circ F$

Note relation to extension principle

Crisp Reasoning

Modus Ponens:

Fact: x is A
Rule: IF x is A THEN y is B
Conclusion: y is B

Fuzzy Reasoning

Single rule with single antecedent

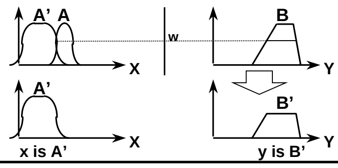
- Fact: x is A'
- Rule: if x is A then y is B
- Conclusion: y is B'
- (Generalized Modus Ponens)

Graphic Representation:

FR - Single Rule, Single Antecedent

Graphical Representation:

- find degree of match w between $\mu_A(x)$ and $\mu_{A'}(x)$
intuitively: degree of belief for antecedent which gets propagated; result should be not greater than w



Fuzzy Reasoning: Single Antecedent

Let A, A', and B be fuzzy sets of X, X, and Y, respectively.

Assumption: the fuzzy implication A->B is expressed as a fuzzy relation R on XxY

The fuzzy set B induced by

fact: x is A' and

premise: IF x is A then y is B

$$\mu_{B'}(y) = \max_x \min\left[\mu_{A'}(x), \mu_R(x,y)\right]$$
$$= \vee_x \left[\mu_{A'}(x) \wedge \mu_R(x,y)\right]$$

or: $B' = A' \circ R = A' \circ (A \to B)$

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Fuzzy Reasoning

Single rule with multiple antecedents

Facts: x is A' and y is B'

Rule: if x is A and y is B then z is C

Conclusion: z is C'

$$\mu_{C'}(z) = \vee_{x,y}\left[\mu_{A'}(x) \wedge \mu_{B'}(y)\right] \wedge \left[\mu_A(x) \wedge \mu_B(y) \wedge \mu_C(z)\right]$$
$$= \vee_{x,y}\left[\mu_{A'}(x) \wedge \mu_{B'}(y) \wedge \mu_A(x) \wedge \mu_B(y)\right] \wedge \mu_C(z)$$
$$= \underbrace{\left\{\vee_x \left[\mu_{A'}(x) \wedge \mu_A(x)\right]\right\}}_{w_1} \wedge \underbrace{\left\{\vee_x \left[\mu_{B'}(y) \wedge \mu_B(y)\right]\right\}}_{w_2} \wedge \mu_C(z)$$

$$C' = \left[A' \circ (A \to C)\right] \cap \left[B' \circ (B \to C)\right]$$

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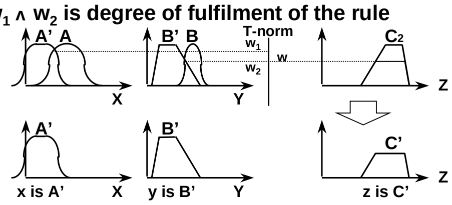
FR - Single Rule, Multiple Antecedents

Graphical Representation

- w_1 denotes degree of compatibility between A and A' and w_2 between B and B'
- $w_1 \wedge w_2$ is degree of fulfilment of the rule



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Fuzzy Reasoning

Multiple rules with multiple antecedent

Fact: x is A' and y is B'

Rule 1: if x is A_1 and y is B_1 then z is C_1

Rule 2: if x is A_2 and y is B_2 then z is C_2

Conclusion: z is C'

Graphic Representation: (next slide)

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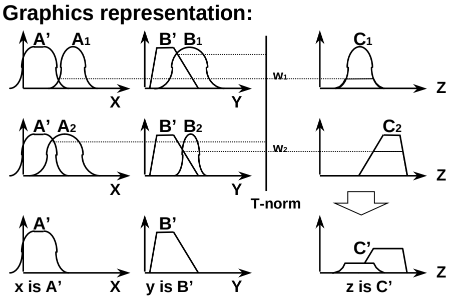
Fuzzy Reasoning

Graphics representation:



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last slide

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- n-ary fuzzy relations: Fuzzy Relations
- fuzzy domains to fuzzy domains: Fuzzy Inference (fuzzy rules, compositional rules of inference)

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- Fuzzy IF-THEN rules
- Compositional rule of inference
- Fuzzy reasoning

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Extension Principle

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$$A = \mu_A(x_1)/x_1 + \mu_A(x_2)/x_2 + \cdots + \mu_A(x_n)/x_n$$

The image of *A* under *f( )* is a fuzzy set *B:*, i.e., *B=f(A)*

$$B = \mu_B(x_1)/y_1 + \mu_B(x_2)/y_2 + \cdots + \mu_B(x_n)/y_n$$

where $y_i = f(x_i)$, $i = 1$ to n.

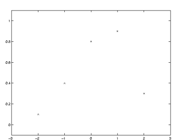
If *f( )* is a many-to-one mapping, then

$$\mu_B(y) = \max_{x=f^{-1}(y)} \mu_A(x)$$

4

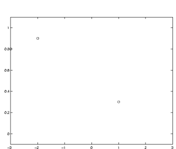
Extension Principle: Example

A=0.1/-2+0.4/-1+0.8/0+0.9/1+0.3/2



f(x)=x^2-3

B=0.1/1+0.4/-2+0.8/-3+0.9/-2+0.3/1
=0.8/-3+(0.4v0.9)/-2+(0.1v0.3)/1
=0.8/-3+0.9/-2+0.3/1



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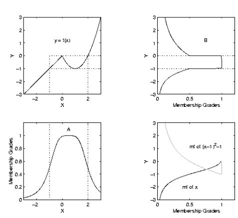
Extension Principle, continuous vars.

Let $\mu_A(x) = bell(x;1.5,2,0.5)$

and $f(x) = \begin{cases} (x-1)^2 - 1 & \text{if } x \ge 0 \\ x & \text{if } x < 0 \end{cases}$



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Fuzzy Relations

A fuzzy relation *R* is a 2D MF:

$$R = \{((x, y), \mu_R(x, y)) | (x, y) \in X \times Y\}$$

Examples:

- x depends on y (x and y are events)
- If x is large, then y is small (x is an observed reading and Y is a corresponding action)
- y is much greater than x (x and y are numbers)

$$\mu_R(x, y) = \begin{cases} \dfrac{y - x}{x + y + 2} & \text{if y} \geq \text{x} \\ 0 & \text{if y} < \text{x} \end{cases}$$

If X={3,4,5} and Y={3,4,5,6,7}

- Express fuzzy relation as a relation matrix

$$R = \begin{bmatrix} 0 & 0.11 & 0.20 & 0.27 & 0.33 \\ 0 & 0 & 0.09 & 0.17 & 0.23 \\ 0 & 0 & 0 & 0.08 & 0.14 \end{bmatrix}$$

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Max-Min Composition

The max-min composition of two fuzzy relations R_1 (defined on *X* and *Y*) and R_2 (defined on *Y* and *Z*) is

$$\mu_{R_1 \circ R_2}(x, z) = \vee_y [\mu_{R_1}(x, y) \wedge \mu_{R_2}(y, z)]$$

Note: calculation very similar to matrix multiplication

Properties:

- Associativity: $R \circ (S \circ T) = (R \circ S) \circ T$
- Distributivity over union:
  $R \circ (S \cup T) = (R \circ S) \cup (R \circ T)$
- Week distributivity over intersection:
  $R \circ (S \cap T) \subseteq (R \circ S) \cap (R \circ T)$
- Monotonicity:
  $S \subseteq T \Rightarrow (R \circ S) \subseteq (R \circ T)$

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Max-min Composition: Example

Let

R_1="x is relevant to y"

R_2="y is relevant to z"

where

$$R_1 = \begin{bmatrix} 0.1 & 0.3 & 0.5 & 0.7 \\ 0.4 & 0.2 & 0.8 & 0.9 \\ 0.6 & 0.8 & 0.3 & 0.2 \end{bmatrix} \quad R_2 = \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.3 \\ 0.5 & 0.6 \\ 0.7 & 0.2 \end{bmatrix}$$

X={ 1 2 3}

Y={α β χ δ} ***Z={a b}***

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Max-min Composition: Example

Calculate:

2 is relevant to a

$$R_1 = \begin{bmatrix} 0.1 & 0.3 & 0.5 & 0.7 \\ 0.4 & 0.2 & 0.8 & 0.9 \\ 0.6 & 0.8 & 0.3 & 0.2 \end{bmatrix} \quad R_2 = \begin{bmatrix} 0.9 & 0.1 \\ 0.2 & 0.3 \\ 0.5 & 0.6 \\ 0.7 & 0.2 \end{bmatrix}$$

X={ 1 2 3}

Y={α β χ δ} ***Z={a b}***

$\mu_{R_1 \circ R_2}(2, a) =$

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Max-Star Composition

Max-product composition:

$$\mu_{R_1 \circ R_2}(x, z) = \vee_y [\mu_{R_1}(x, y) \mu_{R_2}(y, z)]$$

In general, we have max-* composition:

$$\mu_{R_1 \circ R_2}(x, z) = \vee_y [\mu_{R_1}(x, y) * \mu_{R_2}(y, z)]$$

where * is a T-norm operator.

Even more generally, we have (S-norm)-(T-norm) compositions

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Linguistic Variables (knowledge rep.)

Precision vs. significance

A numerical variable can take on numerical *(Age = 65)* or linguistic values *(Age is old)*

A linguistic value is characterized by the variable name (age) a fuzzy set.

All linguistic values form a term set:

T(age) = {young, not young, very young, ...
middle aged, not middle aged, ...
old, not old, very old, more or less old, ...
not very young and not very old, ...}

The syntactic rule refers to how the linguistic values are generated.

The semantic rule defines the membership value of each linguistic variable.

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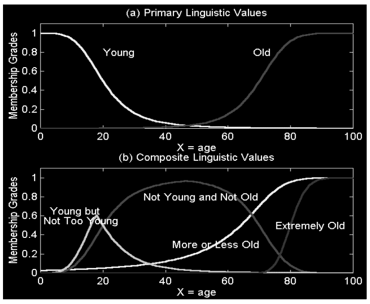
Soft Computing: Fuzzy Rules and Fuzzy Reasoning

Linguistic Values (Terms)



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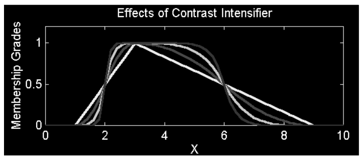
Soft Computing: Fuzzy Rules and Fuzzy Reasoning

Operations on Linguistic Values

Concentration: ⇒ $CON(A) = A^2$

Dilation: ⇒ $DIL(A) = A^{0.5}$

Contrast Intensification: ⇒ $INT(A) = \begin{cases} 2A^2, & 0 \le \mu_A(x) \le 0.5 \\ \neg 2(\neg A)^2, & 0.5 \le \mu_A(x) \le 1 \end{cases}$



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Soft Computing: Fuzzy Rules and Fuzzy Reasoning

Fuzzy IF-THEN Rules

General format:

IF x is A THEN y is B

if <antecedent> then <consequent>

Examples:

- **IF pressure is high THEN volume is small.**
- **IF the road is slippery THEN driving is dangerous.**
- **IF a tomato is red THEN it is ripe.**
- **IF the speed is high THEN apply the brake a little.**

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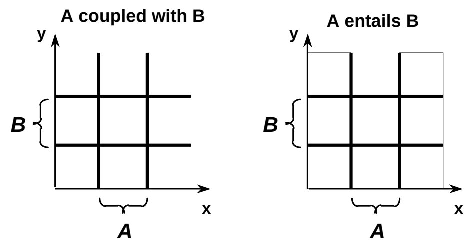
Soft Computing: Fuzzy Rules and Fuzzy Reasoning

Fuzzy IF-THEN Rules

Two ways to interpret "IF x is A THEN y is B":



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Soft Computing: Fuzzy Rules and Fuzzy Reasoning

Fuzzy If-Then Rules

Two ways to interpret "If x is A then y is B":

- **A coupled with B: (*A and B*)**

$$R = A \to B = A \times B = \int \mu_A(x) \tilde{*} \mu_B(y) | (x, y)$$

- **A entails B: (*not A or B*)**
  - Material implication $\neg A \cup B$
  - Propositional calculus $\neg A \cup (A \cap B)$
  - Extended propositional calculus $(\neg A \cap \neg B) \cup B$
  - Generalization of modus ponens

$$\mu_R(x, y) = \sup\left\{c \middle| \mu_A(x) \tilde{*} c \le \mu_B(y) \text{ and } 0 \le c \le 1\right\}$$

- **Note: these all reduce to *not A or B* in two-valued logic**

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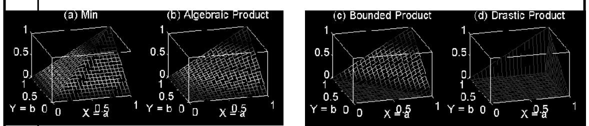
Soft Computing: Fuzzy Rules and Fuzzy Reasoning

Fuzzy IF-THEN Rules

Fuzzy implication function:

$$\mu_R(x, y) = f(\mu_A(x), \mu_B(y)) = f(a, b)$$

A coupled with B



$\mu_R(x, y) = \min(\mu_A(x), \mu_B(y))$ $\mu_R(x, y) = 0 \vee (\mu_A(x) + \mu_B(y) - 1)$

$\mu_R(x, y) = \mu_A(x)\mu_B(y)$ $\mu_R(x, y) = \begin{cases} \mu_A(x) \text{ if } \mu_B(y) = 1 \\ \mu_B(y) \text{ if } \mu_A(x) = 1 \\ 0 \text{ otherwise} \end{cases}$

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Fuzzy IF-THEN Rules

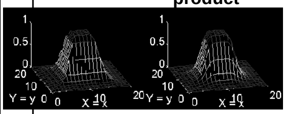
A coupled with B

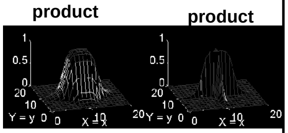
Let

$\mu_A(x) = bell(x;4,3,10)$

$\mu_B(y) = bell(y;4,3,10)$

min | **algebraic product** | **bounded product** | **drastic product**





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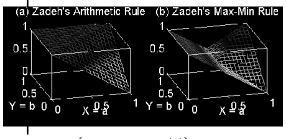
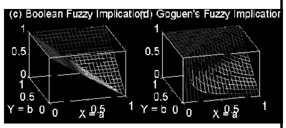
Fuzzy IF-THEN Rules

A entails B

(not an exhaustive list)





$\mu_R(x,y) = \min\left(1, 1-\mu_A(x)+\mu_B(y)\right)$

$\mu_R(x,y) = \max\left(\min\left(\mu_A(x), \mu_B(y)\right), 1-\mu_A(x)\right)$

$\mu_R(x,y) = \min\left(1, \frac{\mu_A(x)}{\mu_B(y)}\right)$

$\mu_R(x,y) = \max\left(1-\mu_A(x), \mu_B(y)\right)$

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Fuzzy IF-THEN Rules

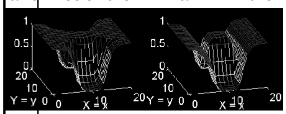
A entails B

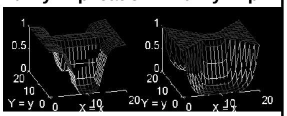
Let

$\mu_A(x) = bell(x;4,3,10)$

$\mu_B(y) = bell(y;4,3,10)$

Zadeh's arithmetic rule | **Zadeh's max-min rule** | **Boolean Fuzzy Implication** | **Gougen's Fuzzy Impl.**





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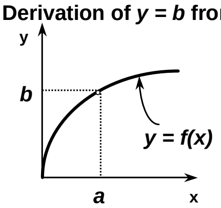
Compositional Rule of Inference

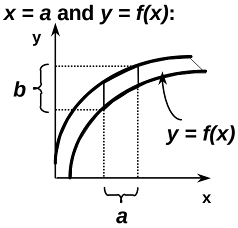
Same idea as max-min composition

Derivation of *y = b* from *x = a* and *y = f(x)*:



a **and** ***b*****: points**

y = f(x) **: a curve**



a **and** ***b*****: intervals**

y = f(x) **: an interval-valued function**

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Compositional Rule of Inference, contd.

To find the resulting interval y=b

(which corresponds to x=a)

- **construct a cylindrical extension of a**
- **find intersection with curve**
- **project intersection to y-axis**

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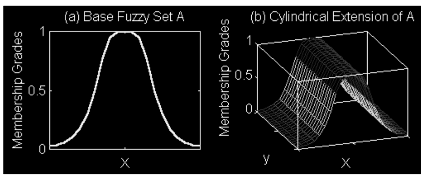
Recall: Cylindrical Extension

Base set A | **Cylindrical Ext. of A**



$\mu_{c(A)}(x,y) = \mu_A(x)$

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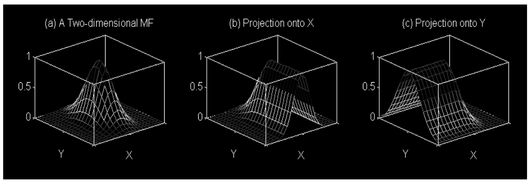
Recall: 2D MF Projection

Two-dimensional MF	Projection onto X	Projection onto Y



$\mu_R(x,y)$	$\mu_A(x) = \max_y \mu_R(x,y)$	$\mu_B(y) = \max_x \mu_R(x,y)$

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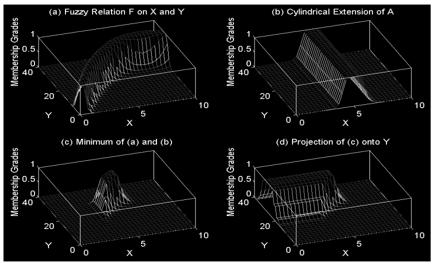
Compositional Rule of Inference

a **is a fuzzy set and** ***y = f(x)*** **is a fuzzy relation:**



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Putting it together: Comp. rule of Inf.

Cylindrical extension with base A

$$\mu_{c(A)}(x,y) = \mu_A(x)$$

Intersection of c(A) with F

$$\mu_{c(A)\cap F}(x,y) = \min\left[\mu_{c(A)}(x,y), \mu_F(x,y)\right] = \min\left[\mu_A(x), \mu_F(x,y)\right]$$

Projection onto y-axis

$$\mu_B(y) = \max_x \min\left[\mu_A(x), \mu_F(x,y)\right] = \vee_x\left[\mu_A(x) \wedge \mu_F(x,y)\right]$$

Representation: $B = A \circ F$

Note relation to extension principle

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Crisp Reasoning

Modus Ponens:

Fact: x is A

Rule: IF x is A THEN y is B

Conclusion: y is B

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Fuzzy Reasoning

Single rule with single antecedent

- Fact: x is A'
- Rule: if x is A then y is B
- Conclusion: y is B'
- (Generalized Modus Ponens)

Graphic Representation:

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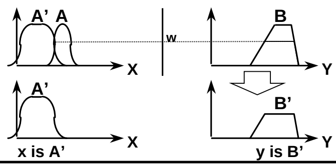
FR - Single Rule, Single Antecedent

Graphical Representation:

- find degree of match w between $\mu_A(x)$ and $\mu_{A'}(x)$
  intuitively: degree of belief for antecedent which gets propagated; result should be not greater than w



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Soft Computing: Fuzzy Rules and Fuzzy Reasoning

Fuzzy Reasoning: Single Antecedent

Let A, A', and B be fuzzy sets of X, X, and Y, respectively.

Assumption: the fuzzy implication A->B is expressed as a fuzzy relation R on XxY

The fuzzy set B induced by

fact: x is A' and

premise: IF x is A then y is B

$$\mu_{B'}(y) = \max_x \min[\mu_{A'}(x), \mu_R(x,y)]$$
$$= \vee_x [\mu_{A'}(x) \wedge \mu_R(x,y)]$$

or: $B' = A' \circ R = A' \circ (A \rightarrow B)$

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Soft Computing: Fuzzy Rules and Fuzzy Reasoning

Fuzzy Reasoning

Single rule with multiple antecedents

Facts: x is A' and y is B'

Rule: if x is A and y is B then z is C

Conclusion: z is C'

$$\mu_{C'}(z) = \vee_{x,y}[\mu_{A'}(x) \wedge \mu_{B'}(y)] \wedge [\mu_A(x) \wedge \mu_B(y) \wedge \mu_C(z)]$$
$$= \vee_{x,y}[\mu_{A'}(x) \wedge \mu_{B'}(y) \wedge \mu_A(x) \wedge \mu_B(y)] \wedge \mu_C(z)$$
$$= \underbrace{\{\vee_x [\mu_{A'}(x) \wedge \mu_A(x)]\}}_{w_1} \wedge \underbrace{\{\vee_x [\mu_{B'}(y) \wedge \mu_B(y)]\}}_{w_2} \wedge \mu_C(z)$$

$$C' = [A' \circ (A \rightarrow C)] \cap [B' \circ (B \rightarrow C)]$$

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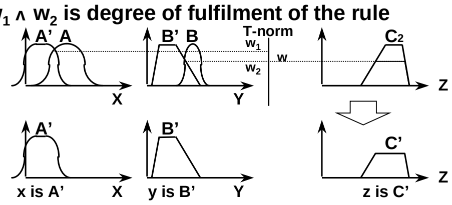
Soft Computing: Fuzzy Rules and Fuzzy Reasoning

FR - Single Rule, Multiple Antecedents

Graphical Representation

- w_1 denotes degree of compatibility between A and A' and w_2 between B and B'
- $w_1 \wedge w_2$ is degree of fulfilment of the rule



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Soft Computing: Fuzzy Rules and Fuzzy Reasoning

Fuzzy Reasoning

Multiple rules with multiple antecedent

Fact: x is A' and y is B'

Rule 1: if x is A_1 and y is B_1 then z is C_1

Rule 2: if x is A_2 and y is B_2 then z is C_2

Conclusion: z is C'

Graphic Representation: (next slide)

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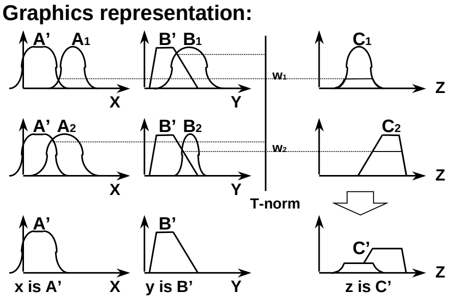
Soft Computing: Fuzzy Rules and Fuzzy Reasoning

Fuzzy Reasoning

Graphics representation:



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Soft Computing: Fuzzy Rules and Fuzzy Reasoning

last slide

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